Valuing Single-period Cash Flows

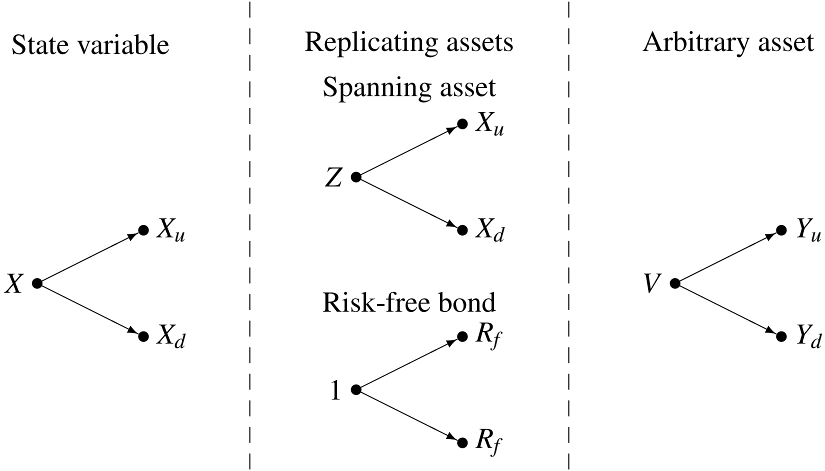
Chapter 3

Real Options in Theory and Practice

Overview

1. Introduce the valuation problem we need to solve
2. Describe the valuation principles that we adopt
3. Use these principles to derive a general valuation formula
4. Describe three separate approaches for implementing this formula:
    - Replication, when the state variable is the price of a traded asset
    - Replication, when suitable forward contracts are traded
    - Capital Asset Pricing Model

Setting the scene



Valuation principles

- Use as much information contained in market prices as possible
- E.g., the law of one price:
  - The prices of any two portfolios that generate identical future cash flows must always be equal
- Our approach:
  - Find a portfolio comprising the risk-free bond and the spanning asset that generates the same cash flow as the asset we are valuing
  - Use the cost of this portfolio as our estimate of the asset's market value

Asset prices and payoffs

- Summary of cash flows for the two alternatives

Transaction	Cost	Payoffs	
		Up state	Down state
Either buy a replicating portfolio			
Buy A bonds	A	AR_f	AR_f
Buy B spanning assets	BZ	BX_u	BX_d
Overall portfolio	$A + BZ$	$AR_f + BX_u$	$AR_f + BX_d$
or buy the asset itself	V	Y_u	Y_d

- Choose A and B so that

$$AR_f + BX_u = Y_u \quad \text{and} \quad AR_f + BX_d = Y_d$$

- The law of one price implies that

$$V = A + BZ$$

Fundamental asset pricing formula

- The result is

$$V = \frac{\pi_u Y_u + \pi_d Y_d}{R_f}, \tag{3.1}$$

where

$$\pi_u = \frac{ZR_f - X_d}{X_u - X_d} \quad \text{and} \quad \pi_d = \frac{X_u - ZR_f}{X_u - X_d} \tag{3.2}$$

- Note that $\pi_u + \pi_d = 1$.
- As long as there are no arbitrage opportunities, π_u and π_d are both greater than zero and less than one.
- π_u and π_d "look like" probabilities.
- We calculate the market value by taking the "expected value" (substituting π_u and π_d in place of the actual probabilities) and discounting using the risk-free interest rate

Risk-neutral probabilities

- Two parallel worlds

	Real world	"Artificial" world
Payoff in an up state	Y_u	Y_u
Payoff in a down state	Y_d	Y_d
Probability of an up move	θ_u	π_u
Probability of a down move	θ_d	π_d
Expected payoff	$\theta_u Y_u + \theta_d Y_d$	$\pi_u Y_u + \pi_d Y_d$
Present value	$\frac{\theta_u Y_u + \theta_d Y_d}{1+RADR}$	$\frac{\pi_u Y_u + \pi_d Y_d}{R_f}$
Arbitrage-free value	$\frac{\pi_u Y_u + \pi_d Y_d}{R_f}$	

- We can calculate the real-world arbitrage-free value of the cash flow by calculating its present value in a risk-neutral world in which up and down moves occur with probabilities π_u and π_d
- We call π_u and π_d risk-neutral probabilities

Adjusting for risk

- The "standard" approach adjusts for risk in the denominator (via the discount rate):

$$V = \frac{\text{Expected cash flow}}{1 + \text{risk-adjusted discount rate}}$$

- Risk high $\Rightarrow$ RADR high $\Rightarrow$ V low
- The "risk-neutral" approach adjusts for risk in the numerator (via the "expected" cash flow):

$$V = \frac{\text{risk-adjusted expected cash flow}}{1 + \text{risk-free interest rate}}$$

- Risk high $\Rightarrow$ Z low $\Rightarrow$ π_u low $\Rightarrow$ "expected" cash flow low $\Rightarrow$ V low

All that remains is to value the spanning asset

- If we know Z then we can consistently estimate the market value of *any* cash flow (X_u, X_d)
- Everything reduces to estimating Z
- We consider three separate approaches:
    - Replication, when X is the price of a traded asset
    - Replication, when suitable forward contracts are traded
    - Capital Asset Pricing Model

Traded asset: Risk-neutral probabilities

- Suppose the state variable is the (cum-dividend) price of a traded asset
  - X is the current price
  - C is the dividend or convenience yield (paid immediately)
  - X_u and X_d are the possible future prices
- Buying one unit of the asset (and receiving the dividend immediately) costs $X - C$. The payoff is the same as the spanning asset's. The law of one price implies that $Z = X - C$.
- Equation (3.2) implies that

$$\pi_u = \frac{ZR_f - X_d}{X_u - X_d} = \frac{(X - C)R_f - X_d}{X_u - X_d} = \frac{\left(1 - \frac{C}{X}\right) R_f - D}{U - D} \qquad (3.3)$$

- If investors require high compensation for risk, then C/X will be large $\Rightarrow \pi_u$ will be small

Traded asset: Example

- The state variable is the spot price of crude oil: $X = 120$, $X_u = 150$, and $X_d = 96$. The convenience yield is 10% of the spot price, so $C = 0.10X = 12$. $R_f = 1.04$.
- An oil field produces one unit of oil, costing $I = 20$
- Cash flows are $Y_u = X_u - I = 130$ and $Y_d = X_d - I = 76$
- Risk-neutral probabilities are

$$\pi_u = \frac{(X - C)R_f - X_d}{X_u - X_d} = \frac{(120 - 12)1.04 - 96}{150 - 96} = 0.3022$$

  and $\pi_d = 1 - \pi_u = 0.6978$
- The oil field is worth

$$V = \frac{\pi_u Y_u + \pi_d Y_d}{R_f} = \frac{0.3022 \times 130 + 0.6978 \times 76}{1.04} = 88.77$$

Forward contracts: Risk-neutral probabilities

- Suppose a forward contract is traded on the state variable
- The forward price is F, so holders of long (short) positions receive (pay)
  - $X_u - F$ if an up move occurs, and
  - $X_d - F$ if a down move occurs
- Taking a long position and buying bonds with a face value of F costs F/R_f. The payoff is the same as the spanning asset's. The law of one price implies that $Z = F/R_f$.
- Equation (3.2) implies that

$$\pi_u = \frac{F - X_d}{X_u - X_d} \quad \text{and} \quad \pi_d = \frac{X_u - F}{X_u - X_d} \tag{3.5}$$

- If investors require high compensation for risk, then F will be small $\Rightarrow \pi_u$ will be small

CAPM: Certainty-equivalent valuation

- Our approach:
  - Find a portfolio comprising the risk-free bond and the market portfolio of risky assets that is "close" to the spanning asset
  - "Close" means that the mean squared tracking error is as small as possible
  - Use the cost of this portfolio as our estimate of the spanning asset's market value
- The estimated market value is

$$Z = \frac{E[\tilde{X}] - (E[\tilde{R}_m] - R_f)\left(\frac{\text{Cov}[\tilde{X},\tilde{R}_m]}{\text{Var}[\tilde{R}_m]}\right)}{R_f} \tag{3.6}$$

This is the *certainty-equivalent form of the CAPM*

CAPM: Risk-neutral probabilities

- Substituting (3.6) into (3.3) shows that the risk-neutral probabilities are

$$\pi_u = \theta_u - \left(\frac{E[\tilde{R}_m] - R_f}{X_u - X_d}\right)\left(\frac{\mathrm{Cov}[\tilde{X}, \tilde{R}_m]}{\mathrm{Var}[\tilde{R}_m]}\right) \qquad (3.6)$$

and

$$\pi_d = \theta_d + \left(\frac{E[\tilde{R}_m] - R_f}{X_u - X_d}\right)\left(\frac{\mathrm{Cov}[\tilde{X}, \tilde{R}_m]}{\mathrm{Var}[\tilde{R}_m]}\right)$$

- If changes in the state variable are positively correlated with the return on the market portfolio then we shift some probability away from the up move to the down move

CAPM: Risk-neutral probabilities (cont.)

- The risk-neutral probabilities can also be written as

$$\pi_u = \frac{K - D}{U - D} \quad \text{and} \quad \pi_d = \frac{U - K}{U - D}, \tag{3.8}$$

where

$$U = X_u / X \quad \text{and} \quad D = X_d / X,$$
$$K = E[\tilde{R}_x] - (E[\tilde{R}_m] - R_f)\beta_x, \tag{3.9}$$

and

$$\beta_x = \frac{\text{Cov}[\tilde{R}_x, \tilde{R}_m]}{\text{Var}[\tilde{R}_m]}$$

is the usual CAPM beta applied to $\tilde{R}_x = \tilde{X}/X$, the proportional change in the state variable

CAPM: Example

- $X = 100$, $X_u = 125$, $X_d = 80$, $\theta_u = \theta_d = 0.5$, $E[\tilde{R}_m] = 1.15$, $R_f = 1.05$, and
$$\frac{\text{Cov}[\tilde{X}, \tilde{R}_m]}{\text{Var}[\tilde{R}_m]} = 50$$
- Price of spanning asset:
$$\begin{aligned} Z &= \frac{E[\tilde{X}] - (E[\tilde{R}_m] - R_f)\left(\frac{\text{Cov}[\tilde{X},\tilde{R}_m]}{\text{Var}[\tilde{R}_m]}\right)}{R_f} \\ &= \frac{0.5 \times 125 + 0.5 \times 80 - (1.15 - 1.05)50}{1.05} \\ &= 92.86 \end{aligned}$$
- Risk-neutral probability:
$$\pi_u = \frac{ZR_f - X_d}{X_u - X_d} = \frac{92.86 \times 1.05 - 80}{125 - 80} = 0.3889$$

CAPM: Example (cont.)

- Alternatively, calculate β_x:

$$\beta_x = \frac{\text{Cov}[\tilde{R}_x, \tilde{R}_m]}{\text{Var}[\tilde{R}_m]} = \frac{1}{X}\frac{\text{Cov}[\tilde{X}, \tilde{R}_m]}{\text{Var}[\tilde{R}_m]} = \frac{50}{100} = 0.5$$

- Risk-adjusted growth factor:

$$\begin{aligned} K &= E[\tilde{R}_x] - (E[\tilde{R}_m] - R_f)\beta_x \\ &= 0.5 \times \frac{125}{100} + 0.5 \times \frac{80}{100} - (1.15 - 1.05)0.5 \\ &= 0.975 \end{aligned}$$

- Risk-neutral probability:

$$\pi_u = \frac{K - D}{U - D} = \frac{0.975 - 0.800}{1.250 - 0.800} = 0.3889$$

Chapter 3: Key concepts

We need to carry out the following steps when estimating the market value of a cash flow to be received after one period

1. Specify a suitable state variable and estimate X, X_u, and X_d
2. Use data on observed bond prices to estimate R_f
3. Calculate the risk-neutral probabilities
   - If the state variable is the price of a traded asset then use (3.3) and (3.4) to calculate π_u and π_d
   - If forwards or futures on the state variable are traded then use (3.5) to calculate π_u and π_d
   - Otherwise, estimate K using (3.9) and then use (3.8) to calculate π_u and π_d
4. Calculate the cash flows, Y_u and Y_d, that we receive after one period
5. Calculate the estimated market value using (3.1)