House Prices, Development Costs, and the Value of Waiting*

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Abstract

This paper demonstrates that new house prices can exceed direct development costs by considerable margins in competitive housing markets with finite price-elasticities of demand and no restrictive land-use regulation. The premium reflects the value of the option to delay developing the marginal piece of undeveloped land. Competition amongst landowners reduces the option value relative to the standard open-city framework, but—as long as undeveloped land is heterogeneous—does not reduce it to zero. Calibrating a special case of the model to U.S. data suggests that the premium is economically significant. In addition to proving that prices can exceed costs without regulation, this paper shows that the relationship between volatility and the rate of investment is more complicated than previously thought.

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1 Introduction

Converting farmland into housing is costly to reverse and the return on development is uncertain. The combination of irreversibility and uncertainty has been shown to make the real option to delay development valuable in a variety of investment situations.[1] However, in markets featuring large numbers of small investors subject to market-wide shocks, competition between investors can sometimes eliminate the option value.[2] Despite these conditions applying in housing markets, there is evidence of economically significant option values in undeveloped-land prices. For example, Plantinga et al. (2002) estimate that the development option comprised 9% of agricultural land value across the US in 1997, with the proportion ranging from 0% in North and South Dakota to 82% in New Jersey. It seems that competition has not eliminated option values

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[1]See Dixit and Pindyck (1994) and Guthrie (2009).

[2]For example, Grenadier (2002) shows that when firms are identical and there are no restrictions on development policies, competition eliminates the option value.

in housing markets. This paper uses an equilibrium model of urban development to investigate the impact that competition has on the option value of undeveloped land.

Urban development has been extensively investigated using the open-city model, in which migration is costless and rents adjust so that households enjoy the same exogenous utility level regardless of their location. The demand for housing in an open city is perfectly price-elastic, since even a slight change from the equilibrium rent level will result in either complete outward migration or infinite inward migration. Capozza and Helsley (1990) analyze the role of real options in an open city, presenting a model of development timing in which land is heterogeneous (with land closer to the CBD generating higher rental income than land further out).[3] They find that the option value of undeveloped land is a positive-valued decreasing function of its distance from the CBD.

Although the open-city framework is convenient, the assumption of costless migration leads to some unrealistic behavior. For example, in the model of Capozza and Helsley (1990), the market-clearing rent generated by a piece of developed land is unaffected by the size of the housing stock, so that the rent received by any individual landowner is unaffected by the development policies of other landowners: there is effectively no competition between owners of undeveloped land. Consequently, if a city's population grows very rapidly and then declines to its previous level, house prices will also return to their previous levels. It seems much more realistic for the resulting substantial excess housing stock to lead to lower rents and house prices, even though the population is unchanged from its initial level.[4]

This paper investigates urban development in the more realistic case of a downward-sloping demand curve for household services.[5] The market-clearing rent received by any individual landowner is thus affected by the development policies of other landowners, so that there is effective competition amongst the owners of undeveloped land. This competition has the potential to reduce the option value of undeveloped land. The effect of positive demand shocks on house prices is partially offset by an increased housing stock, but the irreversibility of land development means that negative demand shocks are not offset by reductions in the housing stock. Thus, compared to the case of perfectly price-elastic housing demand, upside price risk is reduced but downside price risk is unaffected, which reduces the option value of undeveloped land.

In the model, each piece of land is characterized by the rate at which it generates the housing services that individuals consume, which combine the benefits of housing structure and

[3] Many authors have extended their analysis. For example, Capozza and Li (1994) allow landowners to choose the intensity, as well as the timing, of land development; Capozza and Sick (1994) examine the comparative statics of systematic and unsystematic risk; Bar-Ilan and Strange (1996) incorporate construction lags; and Glaeser and Gyourko (2004, 2005) allow houses to be unoccupied when demand falls.

[4] See, for example, Glaeser et al. (2008), who consider the supply response to a housing bubble, with short-term construction leading to permanently lower prices once the bubble has burst.

[5] Capozza and Helsley (1989) consider the case of perfectly inelastic demand with deterministic nonnegative population growth. However, as far as I am aware, there is no parallel analysis of stochastic population growth in a closed city.

associated amenities. The rent received by the owner of a piece of developed land is proportional to the quantity of housing services generated. The paper's treatment of housing services is deliberately abstract, and they can represent any characteristic that affects the desirability of living in a particular location. For example, land with attractive views or a comfortable microclimate might generate more housing services than less desirable land. Another possible source of housing services is distance from the center of a monocentric city. If the housing services generated by a house are proportional to the length of time spent in the house, then greater distance from the city center translates into less time spent at home and therefore fewer housing services. For example, an individual living at the city center may be able to consume 14 hours of housing services per day, whereas an individual with a one-hour commute is able to consume only 12.

When the demand curve is downward-sloping, the option value is lower than for the open-city case and the size of the reduction depends on the cross-sectional variation in the level of housing services generated by land: if all land is identical, the option value falls to zero.[6] In the case of identical land, the owner of any piece of undeveloped land can always delay development, but the level of housing services generated by competing land will be the same even after the delay. The owner will therefore never be in a position to receive a positive development payoff since there will always be a competitor able and willing to preempt him. He is thus effectively faced with the choice of investing as soon as the break-even point is reached or never investing.[7] Since a positive development payoff cannot be achieved, the option value of undeveloped land is zero. Competition is not so effective at eliminating option values when there is intra-city variation in the level of housing services generated by land. Now if the owner of a piece of undeveloped land allows others to invest when his break-even point is reached and then waits long enough, his strongest remaining competitor will have a strictly lower potential for generating housing services. The landowner can invest just before this competitor's break-even point is reached and, because his own land is more valuable than the competitor's, receive a strictly positive development payoff. Even after allowing for the discounting of this payoff, this means that one alternative to investing at the break-even point is to wait and receive a strictly positive payoff at some future date. The landowner's strongest competitor at his own break-even point is also able to wait and receive a strictly positive payoff in the future, which weakens the threat of pre-emption. Moreover, the landowner's competitor's competitor can itself wait and receive a strictly positive investment payoff at some future date, which weakens the preemption threat faced by the first competitor, further weakening the preemption threat that it poses for our landowner. This recursive weakening of the threat of preemption is reflected in the equilibrium

[6] In the limiting case of the model that corresponds to an open city, the lack of competition amongst landowners means that the option value of undeveloped land is independent of the extent of heterogeneity.

[7] If there is only a small number of landowners, then each one has the option to wait and invest last, when it will be able to earn a strictly positive payoff. Thus, competition between a small number of owners will not eliminate the value of waiting. See, for example, the duopoly case considered by Grenadier (1996). However, while oligopoly might be a realistic market structure for some commercial real estate markets, it seems much less so for housing markets.

option value of undeveloped land.

Calibrating the model to U.S. housing market data reveals that new house prices can exceed direct development costs by a substantial margin in regions where housing demand is reasonably price-elastic and expected demand growth is high, when interest rates are low, and where there is considerable cross-sectional variation in the potential for undeveloped land to generate housing services. The last three requirements (and arguably the first one) describe the situation characterizing many of the cities where price–cost ratios have taken the highest values in recent years, suggesting that the results of this paper offer one possible explanation for variations in price–cost ratios across the US.

Another explanation for price–cost ratios substantially greater than one is land-use regulation that prevents developers from constructing buildings of a sufficient density to equate marginal construction cost and price. Glaeser et al. (2005), Green et al. (2005), and Saiz (2010) present empirical evidence supporting the view that such regulation helps explain the sluggish supply response to high house prices. This paper's finding that the option value of land can be economically significant shows that high price–cost ratios can arise even without restrictive land-use regulation. If high price–cost ratios are explained by some combination of regulation and development timing options, then only a part of the observed premium of house prices over construction costs can be attributed to land-use restrictions.

The calibration also allows investigation of the determinants of housing investment. I find that higher long-run growth in demand makes development occur more frequently, but has little impact on the rate of development when it occurs. In contrast, more volatile demand means that development will occur less frequently, but when it does occur it will be more rapid. The first effect dominates, but only slightly, so that the long-run rate of growth in the housing stock is relatively insensitive to demand volatility. The results involving volatility, in particular, extend our current understanding of the determinants of investment behavior.

The remainder of the paper is split into three parts. The model is introduced in Section 2 and its general properties described. This involves first deriving a development policy that a welfare-maximizing social planner would choose and then proving that the resulting house prices can be achieved in an equilibrium setting where individual landowners take house prices as given and adopt development policies that maximize the value of their undeveloped land. Section 2 uncovers the qualitative properties of the equilibrium development policy, but it does not give any indication of its quantitative properties. These are investigated in the second part of the paper, which begins by calibrating a special case of the general model. It then assesses the economic significance of development timing options by investigating the magnitude of price–cost ratios needed to trigger development and calculating the impact of these options on house prices themselves. Section 3 also considers the comparative statics of development behavior. The final part of the paper collects together some of the model's implications for empirical research into housing supply, including the need to reassess the role of land-use regulation in explaining high price–cost ratios and the need to reevaluate how real-option-based theories of investment behavior are tested empirically.

2 Solving for equilibrium land prices

2.1 Model set-up

Consumers demand housing services, which combine the benefits of housing structure and associated amenities. The demand for housing services at date t is described by the inverse demand function $R_t = X_t\Psi(H_t)$, where R_t is the price of one unit of housing services, X_t is an exogenous stochastic variable that determines the overall level of demand, H_t is the total amount of housing services consumed, and Ψ is a strictly-decreasing differentiable function. The state variable X_t follows the geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t d\xi_t,$$

for some constants μ and σ. All agents are risk neutral,[8] so that market values can be found by discounting expected cash flows using the risk-free interest rate r, which I assume is constant and satisfies $\mu < r$.[9]

Each piece of land can be developed at any date, which involves the instantaneous construction of an identical structure, requiring constant lump sum expenditure of C per unit of land. Development is irreversible. Undeveloped land generates no income, whereas developed land generates rental income in proportion to the housing services it provides.[10] Land is heterogeneous, with each piece of land being completely described by a unique real number S. The piece of developed land of type S generates a continuous flow of $h(S)$ units of housing services per unit of time, where h is a decreasing differentiable function, so that the label S sorts land from highest to lowest "quality".

It is socially optimal to develop high-quality land before low-quality land.[11] To see why, consider a development policy in which two pieces of land, of type S' and S'', are developed at dates $t' < t''$ despite having $h(S') < h(S'')$. Such a policy cannot possibly be optimal. Suppose, for example, that we perturb this policy by developing the land of type S'' at date t' and the

[8]This assumption can be relaxed by using the risk-neutral process for X in place of the actual process. Assuming risk-neutrality does not alter the qualitative results and is consistent with empirical evidence that house-price risk-premia are negligible (Flavin and Yamashita, 2002).

[9]This condition is more restrictive than is needed for the present value of rental income to be finite, which is the reason for an upper bound on μ. The prospect of future development means that rental income will be expected to grow more slowly than demand, so that higher values of μ are allowed. The precise upper bound depends on factors such as the price elasticity of demand.

[10]The model could be extended to allow undeveloped land to generate income from, for example, employment in agriculture. The main change would be to replace the development cost C with the sum of construction expenditure and the present value of the foregone agricultural income. The market value of undeveloped land would then equal the sum of the present value of perpetual income from agriculture and the market value of the development option.

[11]As Ohls and Pines (1975) demonstrate, this result may not hold if land-use intensity is endogenous. For example, they show that it can be socially optimal to develop low-value land for low-density housing before developing high-value land for higher-density housing or commercial use. This result might also not hold if the potential flow of housing services fluctuates over time due to changes in factors such as the level of inner city crime.

land of type S' at date t''. The total cost is unchanged, since land is still developed at dates t' and t'' and the cost of development is the same for both pieces of land. The flow of housing services before date t' and after date t'' is unaffected. However, the flow of housing services between these two dates is increased. Thus, the perturbed policy leads to higher overall welfare than the original one, which therefore cannot possibly be optimal.

If we restrict attention to policies that develop high-quality land before low-quality land, we can use S_t to jointly denote the amount of land developed on or before date t and the marginal piece of developed land at that date: land of type S has been developed on or before date t if and only if $S \leq S_t$. Thus

$$H(S_t) \equiv \int_0^{S_t} h(S)dS$$

units of housing services are generated in total at date t. The market-clearing price of housing services at date t is therefore $R_t = X_t \Phi(S_t)$, where $\Phi(S) \equiv \Psi(H(S))$.[12] Since developed land of type $\bar{S}$ generates $h(\bar{S})$ units of housing services, it generates rental income of $h(\bar{S})R_t$ at date t.

Two comments on the microfoundations of the model set-up are in order. A household that wishes to consume some amount $\bar{h}$ of housing services can do so by renting the interval of developed land of type $S \in [S_a, S_b]$, for any combination of S_a and S_b that satisfies $\int_{S_a}^{S_b} h(S)dS = \bar{h}$. The amount of land rented, $S_b - S_a$, varies with the location of this interval. For example, the household can occupy a relatively small house in a good area or a larger one in a bad area. The cost will be the same, $\bar{h}R_t$, in both cases. This means that the cost to the household depends only on the total quantity of housing services consumed and not on the location of the house, enabling the demand of housing services to be aggregated into the function Ψ.[13]

The housing-services function h plays an important role in this model. I allow h to be an arbitrary decreasing differentiable function throughout this section, but considering a specific example illustrates the housing-services concept and the source of heterogeneity in land. In this example, housing services represent the amount of time that an individual is able to spend in his house. The city is monocentric, individuals living at the city center can spend T hours per day in their houses, and commuting time is proportional to distance from the city center. The amount of land within z hours of the city center is

$$S = \int_0^z 2\pi x dx = \pi z^2$$

and the total number of occupancy-hours in this region is

$$H = \int_0^z 2\pi x(T - x)dx = \pi z^2 T - \frac{2}{3}\pi z^3.$$

[12] That is, rent equilibrates the housing market. In order to keep the model tractable, I ignore the vacancies and stickiness in rents that are observed in practice.

[13] Aggregation is justified in the standard monocentric-city framework by a different mechanism: individuals are typically assumed to have utility functions that are linear in consumption of housing and a composite commodity, and rents (net of commuting costs) are the same for all locations. This means that individuals are indifferent about which location they live in, so that the demand for housing can be measured by the number of household groups that wish to live within the city boundary.

Eliminating z between these two equations shows that the total number of occupancy-hours from the S highest-quality houses is

$$H(S) = ST - \frac{2}{3}\sqrt{S^3/\pi}.$$

It follows that the housing-services function h for this particular example is

$$h(S) = H'(S) = T - \sqrt{S/\pi}.$$

2.2 Limiting case: Perfectly elastic demand

I begin by considering the limiting case, corresponding to the open-city framework, in which demand for housing services is perfectly elastic. If the inverse demand function is $\Psi(H) = \Psi_0$ for some constant Ψ_0, then rent is exogenous and equal to $R_t = \Psi_0 X_t$. Developed land of type $\bar{S}$ generates a flow of rent equal to $\Psi_0 h(\bar{S}) X_t$ per unit of time. Thus, the market-clearing rent generated by a piece of land is determined only by the housing services it generates and a city-wide shock that reflects economic conditions; it is independent of the size of the housing stock. The land's market value equals the present value of this flow,

$$V^{\text{p.e.}}(X_t; \bar{S}) = \frac{\Psi_0 h(\bar{S}) X_t}{r - \mu}.$$

This also depends only on the housing services the land generates and the demand shock.

Undeveloped land of type $\bar{S}$ offers a perpetual option to pay C and develop the land. If the option is exercised at date T then the owner's payoff is $V^{\text{p.e.}}(X_T; \bar{S}) - C$. McDonald and Siegel (1986) consider an equivalent situation and show that the option should be exercised as soon as X_t reaches the threshold

$$\hat{X}^{\text{p.e.}}(\bar{S}) = (r - \mu)\left(\frac{\beta}{\beta - 1}\right)\frac{C}{\Psi_0 h(\bar{S})},$$

where

$$\beta = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\frac{2r}{\sigma^2} + \left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2} > 1. \tag{1}$$

Equivalently, it should be exercised as soon as the ratio of the price of the developed land to the development cost reaches the threshold

$$\frac{V^{\text{p.e.}}(\hat{X}^{\text{p.e.}}(\bar{S}); \bar{S})}{C} = \frac{\beta}{\beta - 1} > 1.$$

McDonald and Siegel also show that the market value of the development option (that is, of the undeveloped land) at date t is

$$L(S_t, X_t; \bar{S}) = \frac{C}{\beta - 1}\left(\frac{X_t}{\hat{X}^{\text{p.e.}}(\bar{S})}\right)^{\beta} \tag{2}$$

if $X_t \leq \hat{X}^{\text{p.e.}}(S_t)$. Equation (2) shows that land does not need to be heterogeneous in the open-city framework in order for the option value of undeveloped land to be positive.[14] The

[14] That is, L is positive even if $h(\bar{S})$ is constant for all $\bar{S}$.

remainder of this section considers the situation when the price elasticity of demand for housing services is finite. It will reveal that, in this case, land heterogeneity is a necessary condition for positive option values of undeveloped land.

2.3 Solving the social planner's problem

I am most interested in the house prices that result from the competitive interaction between price-taking owners of undeveloped land when the price elasticity of demand for housing services is finite. This is much more difficult to analyze than the perfectly-elastic case considered in Section 2.2 because the market-clearing rent level and the development policy must be determined simultaneously.

A simple way to determine an equilibrium development policy is to first consider the problem facing a hypothetical social planner who chooses a policy that maximizes the present value of the flow of total surplus. This flow equals the amount by which the surplus associated with housing services exceeds net expenditure on land development, where the former equals

$$X_t \int_0^{S_t} \Phi(S)h(S)dS \tag{3}$$

at date t.[15] In order that this approach generates a competitive equilibrium house price, I assume that the planner uses the risk-free interest rate when calculating this present value. Therefore, the present value at date t of the future flow of total surplus (which is denoted by W) satisfies

$$W(S_t, X_t) = \left(X_t \int_0^{S_t} \Phi(S)h(S)dS \right) dt + e^{-r\,dt} E[W(S_t, X_t + dX_t)]$$

if the housing stock S_t is held constant over the next dt units of time. The cost of increasing the housing stock from S to S^* units equals $C(S^* - S)$, so that the present value at date t of the future flow of total surplus equals

$$W(S_t, X_t) = W(S_t^*, X_t) - C(S_t^* - S_t)$$

if the housing stock is increased to S_t^* units.

At each date t the planner chooses how much land to develop over the next instant of time. Because undeveloped land is ordered from highest to lowest quality, the planner effectively chooses the quantity of developed land, S_t, at each date. The only constraint that the planner faces is that, due to the irreversibility of development, this quantity can never decrease. There are no fixed development costs, the marginal cost of development is constant, and the flow of total surplus (in equation (3)) is concave in the level of the housing stock. I can therefore restrict attention to policies of barrier control, since the best policy of this type achieves an overall level of welfare that is greater than or equal to that resulting from any other land-development

[15] I am ignoring the possibility of externalities created by (for example) population density. While such externalities would be important if the goal here was welfare analysis, my aim is simply to derive a competitive equilibrium development policy.

policy.[16] Each barrier control policy is completely determined by some function $\hat{X}(.)$: the stock of developed land is held constant if $X_t \leq \hat{X}(S_t)$, otherwise it is raised to the smallest value of S^* such that $X_t = \hat{X}(S^*)$. Adoption of such a policy means that the stock of developed land will remain constant for long periods of time, but once the demand variable rises to a sufficiently high level the housing stock will keep increasing by just enough to keep the demand variable at the threshold level $\hat{X}(S_t^*)$.

The first proposition gives a policy function that maximizes the planner's objective function.[17]

Proposition 1 (Socially-optimal development policy) *The planner's objective function is maximized if*

$$\hat{X}(S) = (r - \mu)\left(\frac{\beta}{\beta - 1}\right)\frac{C}{\Phi(S)h(S)}. \tag{4}$$

Rearranging the formula for the socially-optimal policy function, equation (4), gives

$$\frac{\hat{X}(S)\Phi(S)h(S)}{r - \mu} = \left(\frac{\beta}{\beta - 1}\right)C. \tag{5}$$

The left-hand side of equation (5) is the present value of the flow of rent from the marginal piece of developed land in the event that X_t currently equals $\hat{X}(S)$ and the housing stock is permanently held constant at S. The right-hand side is the product of the marginal cost of developing land and the familiar "real option multiplier" $\beta/(\beta - 1)$. It is not immediately clear how the socially-optimal rule compares to a "zero NPV" rule of developing only when the market value of developed land exceeds the development cost: the socially-optimal rule compares an inflated market value (by ignoring future declines in rent due to new development) and an inflated development cost (by applying the option multiplier). The comparison will be clarified in the following subsection, which derives the price of developed land that is implied by the development policy in Proposition 1.

2.4 Solving for the market value of developed land

Let $V(S_t, X_t; \bar{S})$ denote the market value of one unit of developed land (that is, the "house price") of type $\bar{S}$ at date t. It equals the present value of the rental income generated by the property, allowing for future fluctuations in X_t and additions to the housing stock. The following proposition gives the house price at date t when the housing stock evolves according to the socially-optimal rule described in Proposition 1.

[16] See Dixit (1993) for an introduction to barrier control policies and Stokey (2008) for a more detailed discussion. The optimality of a barrier control policy for a mathematically-equivalent problem is proven on pp. 360–367 of Dixit and Pindyck (1994).

[17] All proofs can be found in the appendix.

Proposition 2 (House price) *If $X_t \leq \hat{X}(S_t)$, the market value at date t of developed land of type $\bar{S}$ is*

$$V(S_t, X_t; \bar{S}) = \frac{X_t \Phi(S_t) h(\bar{S})}{r-\mu} \left(1 + \int_{S_t}^{\infty} \frac{\Phi'(S)}{\Phi(S_t)} \left(\frac{X_t}{\hat{X}(S)} \right)^{\beta-1} dS \right). \tag{6}$$

The first term in the expression for the house price, $X_t \Phi(S_t) h(\bar{S})/(r-\mu)$, is the present value of the flow of rent assuming that the housing stock remains at its current level indefinitely. The second term adjusts for the possibility that the housing stock will increase in the future and thereby reduce rental income.[18]

Now that the house price has been determined, it is possible to calculate the price–cost ratio. The following proposition describes the behavior of this ratio for the marginal piece of developed land.[19]

Proposition 3 (Price-cost ratio) *At date t, the price–cost ratio for the marginal piece of developed land satisfies*

$$\frac{V(S_t, X_t; S_t)}{C} \leq 1 + \int_{S_t}^{\infty} \left(\frac{\Phi(S)}{\Phi(S_t)} \right)^{\beta} \left(\frac{h(S)}{h(S_t)} \right)^{\beta-1} \left(\frac{-h'(S)}{h(S)} \right) dS, \tag{7}$$

with equality holding if and only if land is currently being developed.

Since the housing-services function h is decreasing (but not necessarily strictly decreasing), the integral in (7) will be positive unless $h'(S) = 0$ for all land of type $S > S_t$. Therefore, the marginal price–cost ratio will exceed one when land is being developed unless all remaining undeveloped land has the same potential to generate housing services. In other words, the marginal price–cost ratio can exceed one as long as there is some heterogeneity in the quality of the remaining undeveloped land. The implications of this result for the price of undeveloped land are considered in the following subsection.

2.5 Solving for the value of undeveloped land

Since undeveloped land generates no income, the market value of such land is the present value of the development payoff, $V(S_T, X_T; \bar{S}) - C$, where $V(S_T, X_T; \bar{S})$ is the house price at the development date. That is, the value at date t for a unit of undeveloped land of type $\bar{S}$ equals

$$L(S_t, X_t; \bar{S}) = E_t \left[e^{-r(\tilde{T}-t)} \left(V(S_{\tilde{T}}, X_{\tilde{T}}; \bar{S}) - C \right) \right],$$

where the expectation is taken over the uncertain development date $\tilde{T}$.

[18] Note that the downward slope of the demand curve ensures that $\Phi'(S) < 0$, which implies that the second term is less than one. That is, the adjustment reduces house prices relative to the case where the housing stock is held constant indefinitely.

[19] Existing houses generate more housing services than the marginal piece of land, so that—because each house's price is proportional to its flow of housing services—the average price–cost ratio will typically exceed the marginal value. The only exception is when all developed land is identical, in which case the average price–cost ratio equals the marginal value.

Suppose that the owner of a piece of undeveloped land of type $\bar{S}$ follows the socially-optimal development policy in Proposition 1. Since the land is currently undeveloped, it must be the case that X_t is below the threshold at which land of this type would be developed; that is, $X_t < \hat{X}(\bar{S})$. The value of this piece of land equals the present value of receiving a lump sum payment of $V(\bar{S}, \hat{X}(\bar{S}); \bar{S}) - C$ as soon as X_t reaches the (constant) threshold $\hat{X}(\bar{S})$. The next proposition gives the market value of this cash flow stream.

Proposition 4 (Value of undeveloped land) *If $X_t \leq \hat{X}(S_t)$, the market value at date t of undeveloped land of type $\bar{S} > S_t$ is*

$$L(S_t, X_t; \bar{S}) = C \left(\frac{X_t}{\hat{X}(\bar{S})} \right)^{\beta} \int_{\bar{S}}^{\infty} \left(\frac{\Phi(S)}{\Phi(\bar{S})} \right)^{\beta} \left(\frac{h(S)}{h(\bar{S})} \right)^{\beta - 1} \left(\frac{-h'(S)}{h(S)} \right) dS. \tag{8}$$

Equation (8) shows that the value of undeveloped land of type $\bar{S}$ will be positive unless $h'(S) = 0$ for all $S > \bar{S}$, in which case the value is zero. That is, a piece of undeveloped land has positive value unless all remaining undeveloped land is capable of generating an identical flow of housing services.

Equation (5) can be used to write (8) in the form

$$L(S_t, X_t; \bar{S}) = C \left(\frac{\beta - 1}{\beta} \cdot \frac{X_t}{(r - \mu)C} \right)^{\beta} h(\bar{S}) \int_{\bar{S}}^{\infty} (\Phi(S))^{\beta} (h(S))^{\beta - 1} \left(\frac{-h'(S)}{h(S)} \right) dS.$$

Differentiating this equation with respect to $\bar{S}$ shows that

$$\frac{\partial L(S_t, X_t; \bar{S})}{\partial \bar{S}} = \frac{h'(\bar{S})}{h(\bar{S})} \left(L(S_t, X_t; \bar{S}) + C \left(\frac{\beta - 1}{\beta} \cdot \frac{X_t \Phi(\bar{S}) h(\bar{S})}{(r - \mu)C} \right)^{\beta} \right). \tag{9}$$

Thus, the value of undeveloped land is a strictly-decreasing function of $\bar{S}$ wherever the housing-services function is strictly decreasing in $\bar{S}$. In regions where $h'(\bar{S}) = 0$, $\partial L / \partial \bar{S} = 0$. This means that any two pieces of undeveloped land with the same potential to generate housing services will have the same market value. Note that Proposition 1 shows that $\hat{X}(\bar{S})$ is strictly increasing in $\bar{S}$, so that no two pieces of land will ever be developed simultaneously. Thus, even two pieces of identical undeveloped land will be developed at different dates. However, while the piece developed later will generate a higher payoff, the greater discounting will leave the present value unchanged.

2.6 Competitive equilibrium house and land prices

The discussion to this point has centered on identifying a development policy that maximizes the present value of the flow of total surplus. The following proposition shows that this policy, described in Proposition 1, is compatible with a competitive equilibrium in which the owner of each piece of undeveloped land is a price-taker and chooses a development policy that maximizes the market value of that piece of land.

Proposition 5 (A competitive housing market) *There exists a competitive equilibrium in which (i) the owner of each piece of undeveloped land of type $\bar{S} > S_t$ takes the house price as* *given and develops his land as soon as* $X_t \geq \hat{X}(\bar{S})$*; (ii) just enough land is developed at each date to ensure that* $X_t \leq \hat{X}(S_t)$*; (iii) the house price is given by equation (6); and (iv) the value of undeveloped land is given by equation (8).*

The combination of Proposition 4 and part (iv) of Proposition 5 shows that competition between developers eliminates the value of the delay option only when all remaining undeveloped land has the same potential to generate housing services. The owners of all pieces of undeveloped land that would be more attractive once developed than the least attractive piece of land possess a valuable timing option. Land heterogeneity implies a natural development order, with high-quality land being developed before low-quality land. This natural development order prevents competition from eliminating option value, even competition amongst the owners of pieces of identical land. Due to the inherent heterogeneity in undeveloped land, the natural development ordering is to be expected in real estate markets (perhaps more so than in the markets for capital investments that are the normal domain of real options analysis), suggesting that price–cost ratios greater than one are unlikely to be eliminated by competition in real estate markets.

To better understand the role of competition, suppose that all remaining undeveloped land is identical. The owner of any piece of undeveloped land can always delay development, but the level of housing services generated by competing land will be the same even after the delay. The owner will therefore never be in a position to receive a positive development payoff since there will always be a competitor able and willing to preempt him. He is thus effectively faced with the choice of investing as soon as the break-even point is reached or never investing. Since a positive development payoff cannot be achieved, the option value of undeveloped land is zero.

Now suppose that there is some heterogeneity in the remaining undeveloped land. If the owner of a piece of undeveloped land allows others to invest when his break-even point is reached and then waits long enough, his strongest remaining competitor will have the potential to generate strictly fewer housing services. The landowner can invest just before this competitor's break-even point is reached and, because his own land is more valuable than the competitor's, receive a strictly positive development payoff. Even after allowing for the discounting of this payoff, this means that one alternative to investing at the break-even point is to wait and receive a strictly positive payoff at some future date. Of course, the landowner's strongest competitor at his own break-even point is also able to wait and receive a strictly positive payoff in the future, which weakens the threat of pre-emption. Moreover, the landowner's competitor's competitor can itself wait and receive a strictly positive investment payoff at some future date, which weakens the preemption threat faced by the first competitor, which further weakens the preemption threat that it poses for our landowner. This recursive weakening of the threat of preemption ultimately leads to the equilibrium option values reported in Proposition 5.

This intuition might seem to break down in those regions where there are substantial quantities of identical undeveloped land—that is, where the housing-services function $h(\cdot)$ has "flat spots". The owners of these pieces of land compete on equal terms to be the first to develop their land and receive the development payoff. It might seem that such competition—by eliminating the natural ordering discussed above—eliminates the delay option value in this region. However, equation (9) shows that any two pieces of identical undeveloped land will have the same market value.[20] Some landowners will develop their land early and receive a relatively low payoff, while others will develop later and receive a higher payoff. The developments are timed so that the present value of the payoffs are the same for all owners of identical pieces of land. Therefore, individual landowners have no incentive to deviate from this sequence once an ordering is achieved, because the present value of their development payoff is independent of their place in the development queue.[21] As a result, even competition between the owners of identical pieces of land will not eliminate the value of the landowners' delay options when there is lower-quality land waiting to be developed.

3 A special case

This section analyzes a specific example of the model developed in Section 2 in order to assess the economic significance of the results derived there. Such an assessment is necessary because if competition amongst landowners reduces the value of development-delay options to economically insignificant levels their mere existence is unimportant.

Increasing the quantity of developed land changes the location of the city boundary, changing the location of the marginal piece of developed land. I suppose that a one percent increase in the quantity of developed land means that a unit of land on the new city boundary generates ε percent fewer housing services than a unit of land on the old boundary, for some nonnegative constant $\varepsilon < 1$.[22] The parameter ε measures the degree of heterogeneity in land: low values of ε indicate that land is relatively homogeneous, whereas high values indicate that it is relatively heterogeneous. I normalize the model so that initially the housing stock is $S_0 = 1$ and the housing services generated by the marginal piece of developed land equals 1; that is, $h(S) = S^{-\varepsilon}$ and the housing-services supply function is $H(S) = S^{1-\varepsilon}/(1-\varepsilon)$. The amount of land available to be developed, and therefore the potential supply of housing services, is unbounded from above.

I also assume that the price elasticity of demand for housing services equals $-\eta$ for some positive constant η, with demand normalized so that the rent generated by the marginal piece of developed land is initially equal to 1.[23] It follows that $\Psi(H) = (1-\varepsilon)^{-1/\eta} H^{-1/\eta}$ for all $H \geq H(S_0)$, and $\Phi(S) = S^{-(1-\varepsilon)/\eta}$ for all $S \geq S_0$. With this choice of demand and housing-services functions, it is easy to evaluate the outputs of the model using the results of the

[20]This follows from the observation that $\partial L/\partial \bar{S} = 0$ whenever $h'(\bar{S}) = 0$.

[21]This development pattern can be achieved by having each landowner play an appropriate mixed strategy. For example, if there are L units of identical land, of which ΔS units must be developed, then divide the land into n identical pieces and suppose that each piece is developed with probability $p = \Delta S/L$. The amount of land developed has mean $np(L/n) = \Delta S$ and variance $np(1-p)(L/n)^2 = \Delta S(L - \Delta S)/n$. In the limit as $n \to \infty$, exactly ΔS units of land will be developed.

[22]The restriction $\varepsilon < 1$ is needed to ensure that the total supply of housing services is finite.

[23]Since the supply of housing services cannot fall, the shape of the demand curve for $H < H_0$ is unimportant. All I require is that the area under the demand curve in this region is finite.

previous section. For instance, substituting them into equation (4) shows that the socially-optimal development threshold is

$$\hat{X}(S) = (r - \mu) \left(\frac{\beta}{\beta - 1} \right) C S^{\varepsilon + (1-\varepsilon)/\eta}. \tag{10}$$

Similarly, Proposition 2 implies that the house price equals

$$V(S_t, X_t; \bar{S}) = h(\bar{S}) \frac{X_t S_t^{-(1-\varepsilon)/\eta}}{r - \mu} \left(1 - \frac{1 - \varepsilon}{\beta(1-\varepsilon) + (\beta - 1)\varepsilon\eta} \left(\frac{X_t}{\hat{X}(S_t)} \right)^{\beta - 1} \right) \tag{11}$$

at date t, while Proposition 4 implies that undeveloped land is worth

$$L(S_t, X_t; \bar{S}) = \frac{\varepsilon\eta C}{\beta(1-\varepsilon) + (\beta - 1)\varepsilon\eta} \left(\frac{X_t}{\hat{X}(\bar{S})} \right)^{\beta}$$

at the same date.

3.1 Calibrating the model

I calibrate the model in this section to U.S. residential real estate data. The parameters of interest are μ, σ, η, ε, and r. Since I consider various scenarios for average demand growth (μ) and the risk-free interest rate (r), just σ, η, and ε remain as parameters to be calibrated.

Volatility (σ) reflects the behavior of the demand driver X, which cannot be observed directly. Therefore I need to infer information from data on observable variables. It is straightforward to show that, as long as $0 < X_t < \hat{X}(S_t)$, house prices have volatility

$$\sigma \frac{X}{V} \frac{\partial V}{\partial X} = \sigma \left(\frac{1 - \frac{\beta(1-\varepsilon)}{\beta(1-\varepsilon)+(\beta-1)\varepsilon\eta} \left(\frac{X_t}{\hat{X}(S_t)} \right)^{\beta-1}}{1 - \frac{1-\varepsilon}{\beta(1-\varepsilon)+(\beta-1)\varepsilon\eta} \left(\frac{X_t}{\hat{X}(S_t)} \right)^{\beta-1}} \right) < \sigma.$$

Thus throughout this region house prices are less volatile than the exogenous state variable: the house price equals the present value of all future rental income, so that positive demand shocks are somewhat dampened by the resulting increased probability of future expansion of the housing stock. House price volatility will tend to underestimate demand volatility in the model. I use the repeat-transaction house price index published by the Federal Housing Finance Agency (FHFA) each quarter, converting the data into real quarterly growth rates using the national CPI excluding shelter and then estimating the annualized volatility for each return series. After restricting the sample to those Metropolitan Statistical Areas (MSA) with at least 80 quarterly observations, I obtain a set of 328 volatility estimates that has a mean of 0.0526 and a standard deviation of 0.0176.[24] In the numerical analysis that follows, I set $\sigma = 0.06$. Recalling that house prices are less volatile than demand in the model, this choice means that a quarter of the MSAs in the FHFA data have estimated volatility that is greater than my chosen value of σ. Thus, setting $\sigma = 0.06$ might be regarded as a conservative choice.

[24]The FHFA also reports annualized volatility estimates of nominal house prices at the state level. In the third quarter of 2008, the set of volatility estimates for all states and Washington DC had a mean of 0.0815 and a standard deviation of 0.0072.

The second parameter, η, determines the demand response to changes in rent: a one percent increase in rent reduces the consumption of housing services by η percent. Empirical estimates of the price elasticity are typically less than one in magnitude. For example, Hanushek and Quigley (1980) analyze the response of selected low-income households in Phoenix and Pittsburgh to rent subsidies and obtain elasticity estimates of -0.45 and -0.64 respectively. I consider a range of values for η, from 0.5 (representative of empirical studies) to infinity (the open-city level).

The remaining parameter, ε, is chosen by setting the price elasticity of housing supply, which is determined endogenously in this model, equal to its empirical counterpart. A conventional (that is, static) housing supply curve can be constructed by tracing out the path of the equilibrium price–quantity combination for different levels of the demand variable, while holding the house type, $\bar{S}$, fixed. Assuming that the housing stock is currently S_0, then for any $X \leq \hat{X}(S_0)$ the irreversibility of development means that this part of the supply curve is perfectly inelastic; that is $Q = S_0$. Higher levels of X trigger land development, with the housing stock increasing to $\hat{X}^{-1}(X)$ and the house price to $V(\hat{X}^{-1}(X), X; \bar{S})$. The implied price–quantity pair is

$$P = h(\bar{S}) \frac{X(\hat{X}^{-1}(X))^{-(1-\varepsilon)/\eta}}{r-\mu} \left(1 - \frac{1-\varepsilon}{\beta(1-\varepsilon)+(\beta-1)\varepsilon\eta}\right), \quad Q = \hat{X}^{-1}(X). \tag{12}$$

Eliminating X between the two expressions in (12) shows that price and quantity are related by

$$Q = \left(\frac{P}{Ch(\bar{S})}\left(1 - \frac{\varepsilon\eta}{\beta(1-\varepsilon+\varepsilon\eta)}\right)\right)^{1/\varepsilon}. \tag{13}$$

In particular, the price elasticity of housing supply is $1/\varepsilon$. Green et al. (2005) use annual data for the period 1979–1996 to estimate price elasticities of supply for 45 MSAs. The median elasticity is 5.25 (Rochester), corresponding to $\varepsilon = 0.19$. The top and bottom quintiles are 11.4 (Denver) and 1.77 (Boston), respectively, corresponding to values for ε of 0.09 and 0.56. I will use these three values in the analysis that follows.

3.2 The price–cost ratio

This section uses the calibrated model to calculate realistic magnitudes of the price–cost ratio that determines the timing of development. The equilibrium development rule has a particularly simple form.

Proposition 6 (Land development policy) *At any point in time, just enough land is developed to keep the price–cost ratio of the marginal piece of developed land less than or equal to* $1/\left(1 - \frac{\varepsilon\eta}{\beta(1-\varepsilon+\varepsilon\eta)}\right)$.

The threshold for the marginal price–cost ratio takes values ranging from 1 to $\beta/(\beta-1)$, depending on the heterogeneity of land and the price elasticity of housing demand. It is an increasing function of ε and (provided $\varepsilon > 0$) an increasing function of η; it is independent of η if $\varepsilon = 0$.[25] Thus, if land is relatively heterogeneous or housing demand is relatively elastic then the marginal price–cost ratio can rise to relatively high levels before development occurs.

[25] Recall the parameter restrictions $\varepsilon < 1$ and $\eta > 0$.

Table 1: Development thresholds for various scenarios

		Low heterogeneity ($\varepsilon = 0.09$)			Medium heterogeneity ($\varepsilon = 0.19$)			High heterogeneity ($\varepsilon = 0.56$)		
η	μ	$r = 0.03$	$r = 0.04$	$r = 0.05$	$r = 0.03$	$r = 0.04$	$r = 0.05$	$r = 0.03$	$r = 0.04$	$r = 0.05$
1/2	−0.02	1.0035	1.0034	1.0034	1.0079	1.0077	1.0075	1.0300	1.0292	1.0284
	0.00	1.0103	1.0091	1.0082	1.0233	1.0204	1.0185	1.0921	1.0802	1.0719
	0.02	1.0338	1.0260	1.0214	1.0785	1.0599	1.0490	1.3691	1.2649	1.2092
1	−0.02	1.0068	1.0066	1.0064	1.0144	1.0140	1.0137	1.0438	1.0425	1.0414
	0.00	1.0199	1.0175	1.0158	1.0430	1.0376	1.0339	1.1382	1.1196	1.1070
	0.02	1.0665	1.0509	1.0417	1.1517	1.1140	1.0923	1.6345	1.4318	1.3318
2	−0.02	1.0125	1.0122	1.0119	1.0245	1.0238	1.0232	1.0568	1.0552	1.0537
	0.00	1.0371	1.0325	1.0293	1.0744	1.0649	1.0583	1.1843	1.1587	1.1414
	0.02	1.1293	1.0976	1.0793	1.2843	1.2077	1.1656	1.9908	1.6304	1.4693
∞	−0.02	1.0809	1.0786	1.0764	1.0809	1.0786	1.0764	1.0809	1.0786	1.0764
	0.00	1.2768	1.2358	1.2086	1.2768	1.2358	1.2086	1.2768	1.2358	1.2086
	0.02	3.2597	2.1671	1.8015	3.2597	2.1671	1.8015	3.2597	2.1671	1.8015

The table reports the development threshold for the marginal price–cost ratio, from Proposition 6, for the indicated parameter values. All calculations assume $\sigma = 0.06$.

In the limiting case when housing demand is perfectly elastic, the market-clearing level of rent is independent of the housing stock: demand shocks affect rent "one-for-one" in this case. In contrast, when the housing demand curve is downward sloping, the effect of positive demand shocks on rent is partially offset by an increased housing stock. However, the irreversibility of land development means that negative demand shocks are not offset by reductions in the housing stock, so that they affect rent one-for-one (as if housing demand is perfectly elastic). Thus, compared to the case of perfectly elastic housing demand, upside rent-risk is reduced but downside rent-risk is unaffected. This reduces the option value of undeveloped land. Since landowners must be compensated for the loss of this option value (as well as the direct development cost) when they develop their land, the downward slope in the housing demand curve lowers the development threshold, consistent with Proposition 6.

It is easily shown that β is an increasing function of r and a decreasing function of μ and σ. Proposition 6 therefore implies that the threshold for the marginal price–cost ratio is a decreasing function of r and an increasing function of μ and σ.[26] Therefore, the marginal price–cost threshold should be especially high in markets characterized by expected rapid demand growth, high demand volatility, and low interest rates. These three factors are all consistent with the value of delay options being relatively high.

In order to give an indication of the economic significance of the extent to which the marginal price–cost ratio can exceed one, Table 1 shows the development threshold for several different scenarios. The table is divided into four panels, each one corresponding to a different demand

[26] In the open-city model of Capozza and Helsley (1990), the development threshold for the price–cost ratio is decreasing in the interest rate and increasing in the drift and volatility of the exogenous rent level.

elasticity; each panel is divided into three parts, each one corresponding to a different degree of land heterogeneity; each part shows the price–cost development threshold for different combinations of μ and r. The top left-hand portion of the table shows that when demand elasticity is low and undeveloped land is reasonably homogeneous, competition amongst landowners prevents the marginal price–cost ratio from climbing significantly above 1. However, the rest of the table shows that such competition does not prevent high marginal price–cost ratios when demand is relatively elastic or undeveloped land has a relatively high degree of heterogeneity. In some cases, the extent to which the ratio can exceed one is modest, but in other cases ratios above 1.5 are possible. The highest ratios occur when housing demand is reasonably elastic, the interest rate is low, expected growth in demand is high, and land is highly heterogeneous. The last three requirements (and arguably the first one) describe the situation characterizing many of the cities where price–cost ratios have taken the highest values in recent years.

3.3 The effect of timing flexibility on house prices

The discussion in Section 3.2 gives an indication of the quantitative significance of delay options' effect on equilibrium land development and, by extension, the value of the real option embedded in undeveloped land. In this section, I concentrate on the quantitative significance of delay options' effect on the equilibrium price of developed land. Of course, in this model, the owner has no further flexibility once land has been developed, so there is no real option premium embedded in the price of developed land. Nevertheless, as I now show, the presence of embedded real options in undeveloped land raises the price of developed land by an amount that is economically significant.

The relevant counterfactual is the case where all owners of undeveloped land ignore the option value of their land when making development decisions; that is, landowners incorrectly infer from the fact that their land generates no income as long as it is undeveloped that it is worthless until it is developed. They therefore simply compare the price of the land when it is developed with the construction cost, and develop the land whenever the former is greater than or equal to the latter. Since they will only ever break even from developing their land, it follows that the value of undeveloped land will always equal zero in this case. Valuing developed land is more complicated because the house price depends on the future flow of rent, which depends on the future path of the housing stock, which is different from the path determined by the policy in Proposition 1. The development threshold and house price function need to be determined simultaneously in such a way that if the housing stock is determined by the development threshold then the marginal price–cost ratio will equal one whenever land is being developed. The calculation must recognize that, as in the main model, developing land is irreversible. The next proposition presents the development threshold and house price function that satisfy these requirements.

Proposition 7 (Counterfactual development policy) *In the special case considered in Section 3, if landowners develop their land whenever the house price is greater than or equal to the* *construction cost then the housing stock is determined by the development threshold*

$$\hat{X}^*(S) = \frac{\hat{X}(S)}{1 + \frac{\varepsilon\eta}{\beta(1-\varepsilon)+(\beta-1)\varepsilon\eta}}$$

and developed land of type $\bar{S}$ is worth

$$V^*(S_t, X_t; \bar{S}) = h(\bar{S}) \frac{X_t S_t^{-(1-\varepsilon)/\eta}}{r-\mu} \left(1 - \frac{1-\varepsilon}{\beta(1-\varepsilon)+(\beta-1)\varepsilon\eta} \left(\frac{X_t}{\hat{X}^*(S_t)}\right)^{\beta-1}\right)$$

whenever $X_t \leq \hat{X}^*(S_t)$.

Notice that the house-price function has the same form as the competitive-equilibrium version in equation (11), except that the term correcting for the impact of future development depends on the counterfactual development threshold, $\hat{X}^*(S_t)$, rather than the competitive-equilibrium one, $\hat{X}(S_t)$.

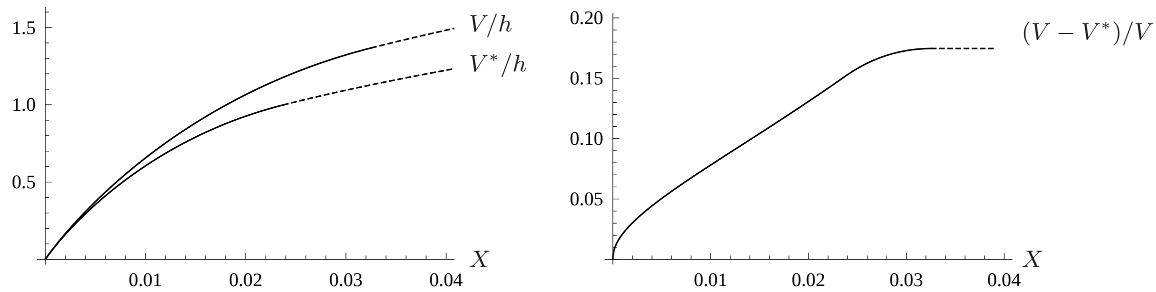
The development threshold is smaller than the competitive-equilibrium threshold by a constant factor that exceeds one whenever $\varepsilon > 0$. Thus, for any given current level of demand and housing stock, future land development will occur sooner if landowners ignore the option value of their land. This reduces the future flow of rent to developed land, thereby lowering house prices.

In order to assess the magnitude of this reduction in house prices, the left-hand graph in Figure 1 plots the house price per unit of housing services ($V(S_t, X_t; \bar{S})/h(\bar{S})$ and $V^*(S_t, X_t; \bar{S})/h(\bar{S})$) as a function of X for a given level of the housing stock in the competitive-equilibrium (the top curve) and counterfactual cases (the bottom curve).[27] The solid portions of the two curves correspond to the situations where $X_t \leq \hat{X}(S_t)$ and $X_t \leq \hat{X}^*(S_t)$; that is, where the housing stock is held constant. The dashed portions of each curve correspond to the case where development is underway: for instance, for the counterfactual situation, the house price assumes that the housing stock is immediately raised to S_t^* such that $X_t = \hat{X}^*(S_t^*)$, which is the level implied by the development threshold $\hat{X}^*$. When demand is very low, the prospect of any expansion of the housing stock in the short-to-medium term is remote, regardless of which of the two development policies is followed. Thus, the house price is similar in both cases. However, the effect of future development becomes evident as demand rises. For moderate levels of X, development will be imminent if landowners ignore their delay options, and less so if they do not. Thus, the house price will be lower if landowners ignore delay options than if they do not.

The right-hand graph in Figure 1 plots the proportional reduction in the house price (as a function of demand) if landowners immediately and permanently ignore the option value of undeveloped land, with the dashed portion corresponding to the region where development would be underway in the competitive-equilibrium case. Consistent with the left-hand graph, the fall in house prices would be negligible if demand is so low that development is unlikely to occur in the medium term in either scenario. However, when demand is high—so that future development

[27] The calculations assume $\varepsilon = 0.56$, $C = 1$, $S_0 = 1$, $r = 0.03$, $\mu = 0.02$, $\sigma = 0.06$, and $\eta = 0.5$.

Figure 1: The effect of development-timing options on the market value of developed land



The left-hand graph plots the house price per unit of housing services as a function of X for a given level of the housing stock in the competitive-equilibrium (the top curve) and counterfactual cases (the bottom curve). The dashed portions of the two curves correspond to demand levels where development is underway. The right-hand graph plots the proportional reduction in the house price if landowners immediately and permanently ignore the option value of undeveloped land, with the dashed portion indicating where development would be underway in the competitive-equilibrium case. All calculations assume $\varepsilon = 0.56$, $C = 1$, $S_0 = 1$, $r = 0.03$, $\mu = 0.02$, $\sigma = 0.06$, and $\eta = 0.5$.

plays a much more important role—the fall in prices would be economically significant. For the parameters used here, approximately 17.5% of a house's value derives from the fact that owners of undeveloped land optimally delay development past the point where the price of a new house exceeds its construction cost.

One measure of the importance of option value for house prices is the proportional reduction in the house price if landowners immediately and permanently ignore the option value of undeveloped land. The next proposition calculates this measure for the special case where $X_t = \hat{X}(S_t)$, which corresponds to the junction of the solid and dashed curves in the right-hand graph in Figure 1. This level of demand is sufficiently high that the housing stock will immediately increase by a finite amount in the case that landowners ignore their delay options, with the housing stock rising to S_t^* defined implicitly by $\hat{X}^*(S_t^*) = \hat{X}(S_t)$.

Proposition 8 ("Option component" in house prices) *In the special case considered in Section 3, the proportional reduction in the house price if landowners immediately and permanently ignore the option value of undeveloped land, for the special case where $X_t = \hat{X}(S_t)$, is*

$$\frac{V(S_t, \hat{X}(S_t); \bar{S}) - V^*(S_t^*, \hat{X}(S_t); \bar{S})}{V(S_t, \hat{X}(S_t); \bar{S})} = 1 - \left(1 + \frac{\varepsilon\eta}{\beta(1-\varepsilon) + (\beta-1)\varepsilon\eta}\right)^{-(1-\varepsilon)/(1-\varepsilon+\varepsilon\eta)}.$$

This quantity is reported in Table 2 using the same format as Table 1. When housing demand is perfectly elastic, the discussion in Section 2.2 shows that equilibrium house prices are independent of the housing stock, and therefore independent of the development policy. This is reflected in the bottom panel of Table 2, which shows that house prices are unaffected if landowners adopt a different development policy. However, the rest of the table shows that

Table 2: Proportion of house prices attributable to the option to delay land development

		Low heterogeneity ($\varepsilon = 0.09$)			Medium heterogeneity ($\varepsilon = 0.19$)			High heterogeneity ($\varepsilon = 0.56$)		
η	μ	$r = 0.03$	$r = 0.04$	$r = 0.05$	$r = 0.03$	$r = 0.04$	$r = 0.05$	$r = 0.03$	$r = 0.04$	$r = 0.05$
0.50	−0.02	0.0034	0.0033	0.0032	0.0070	0.0068	0.0067	0.0179	0.0174	0.0170
	0.00	0.0097	0.0086	0.0078	0.0204	0.0179	0.0162	0.0524	0.0460	0.0416
	0.02	0.0311	0.0242	0.0200	0.0654	0.0508	0.0419	0.1747	0.1338	0.1096
1	−0.02	0.0061	0.0060	0.0058	0.0115	0.0112	0.0109	0.0187	0.0182	0.0177
	0.00	0.0178	0.0156	0.0141	0.0335	0.0295	0.0266	0.0554	0.0485	0.0437
	0.02	0.0569	0.0442	0.0365	0.1081	0.0837	0.0690	0.1944	0.1461	0.1185
2	−0.02	0.0103	0.0101	0.0098	0.0163	0.0159	0.0155	0.0155	0.0150	0.0146
	0.00	0.0300	0.0264	0.0239	0.0477	0.0419	0.0379	0.0466	0.0407	0.0366
	0.02	0.0965	0.0748	0.0617	0.1566	0.1205	0.0990	0.1765	0.1288	0.1029
∞	−0.02	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	0.02	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000

The table reports the proportional reduction in house prices if landowners immediately and permanently ignore the option value of undeveloped land, for the indicated parameter values. All calculations assume $X_t = \hat{X}(S_t)$ and $\sigma = 0.06$.

the “option component” is positive when the demand elasticity is finite. When undeveloped land is reasonably homogeneous ($\varepsilon = 0.09$) the option component is almost worthless, since in this case it is optimal to develop land as soon as the house price is only very slightly greater than the construction cost: that is, removing the delay option from consideration has little effect on behavior. However, the option component can be economically significant when the heterogeneity of undeveloped land is high. For example, in the case where $\varepsilon = 0.56$, the risk-free interest rate is low, and demand is expected to grow rapidly, more than 15% of the price of developed land can be attributed to the option to delay developing undeveloped land.

3.4 Development behavior

The growing empirical literature on the role of real options in real estate has examined the effects of competition and volatility on investment in both commercial and residential real estate. The model in this section makes several predictions regarding these effects, which I now discuss. I focus on two aspects of investment—the frequency and quantity of land development—and how they are affected by the degree of heterogeneity of undeveloped land, average demand growth, and the volatility of demand growth.

Consider an arbitrary interval of time, from date t to $t+T$. Development occurs during this period only if the state variable exceeds the initial development threshold, $\hat{X}(S_t)$, at some point between dates t and $t+T$, in which case the end-of-period housing stock will depend on the maximum level attained by the state variable during that period. Specifically, the end-of-period housing stock, S_{t+T}, will satisfy

$$\hat{X}(S_{t+T}) = \max_{t \le t' \le t+T} X_{t'}.$$

If the maximum level is less than or equal to $\hat{X}(S_t)$ then no development occurs and the end-of-period housing stock is $S_{T+t} = S_t$. The closer the state variable is to the development threshold at the beginning of the period—that is, the closer that $X_t/\hat{X}(S_t)$ is to one—the higher is the probability that some development will occur and the greater is the expected amount of development.

The structure of the model is such that all of the required quantities depend on the long-run distribution of a variable, $X_t/\hat{X}(S_t)$, that is known to evolve according to a geometric Brownian motion with an upper reflecting barrier at 1, drift of μ, and volatility of σ. The following lemma gives the density function for the long-run distribution of this variable.

Lemma 1 (Unconditional distribution of $X/\hat{X}(S)$**)** *Provided that* $\mu > \sigma^2/2$*, the long-run distribution of* $X/\hat{X}(S)$ *is described by the density function*

$$f(\theta) = \left(\frac{2\mu}{\sigma^2} - 1\right) \theta^{2\mu/\sigma^2 - 2}, \quad 0 \le \theta \le 1.$$

The first proposition in this subsection gives the long-run average rate of development over an arbitrary period spanning T units of time.

Proposition 9 (Long-run average change in the log housing stock) *Provided that* $\mu > \sigma^2/2$*, the unconditional expected change in the log housing stock during a period spanning T units of time is*

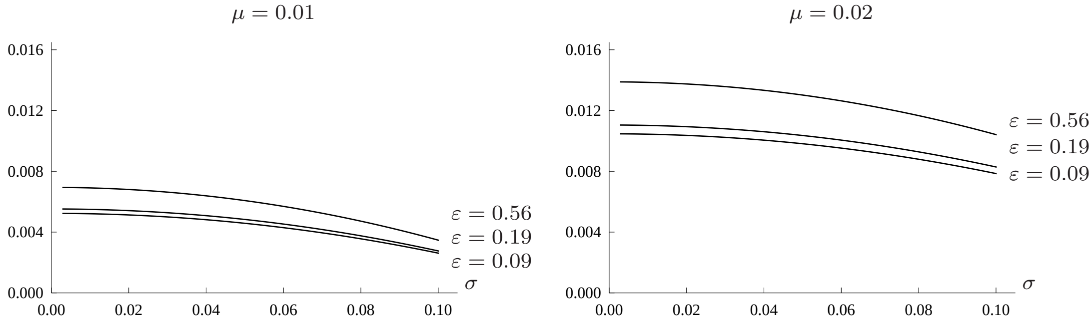
$$E[\log S_{t+T} - \log S_t] = \left(\frac{\left(\mu - \frac{1}{2}\sigma^2\right)\eta}{1 - \varepsilon + \varepsilon\eta}\right) T.$$

The long-run average change in the log housing stock is an increasing function of the average growth rate in demand and a decreasing function of demand volatility. It is an increasing function of η, so that the housing stock will grow more quickly when demand is more elastic. It is an increasing function of ε if $\eta < 1$ and a decreasing function of ε if $\eta > 1$.

The three curves in each graph in Figure 2 plot the long-run average annual change in the log housing stock as a function of demand volatility, assuming that the price elasticity of housing demand is -0.5. The top curve corresponds to the case where $\varepsilon = 0.56$, the middle curve to $\varepsilon = 0.19$, and the bottom one to $\varepsilon = 0.09$. The long-run average growth rate in demand is $\mu = 0.01$ in the left-hand graph and $\mu = 0.02$ in the right-hand graph. Somewhat surprisingly, in view of the conventional real-option wisdom that greater volatility reduces investment, long-run average growth in the housing stock is not particularly sensitive to the level of demand volatility: greater volatility leads to only slightly slower growth on average.

However, a simple measure of development activity such as the long-run average growth rate in the log housing stock hides some important differences between high- and low-volatility regions, as well as between regions with different degrees of land heterogeneity. As described

Figure 2: Long-run average annual change in the log housing stock



The three curves in each graph plot the long-run average annual change in the log housing stock for the indicated parameters. The values of parameters not shown on the graphs are $\eta = 0.5$, $r = 0.03$, and $T = 1$.

above, the model predicts that there will be periods of intense development activity, interrupted by periods of time during which there is no development activity at all. The long-run average growth rate does not adequately capture this behavior. Therefore, the remainder of this section focusses on two measures of development activity: the proportion of the time that development activity is observed, and the average rate of development when it is observed. The next proposition gives the probability that development occurs in an arbitrary period spanning T units of time.

Proposition 10 (Probability of development occurring) *Provided that $\mu > \sigma^2/2$, the unconditional probability that the housing stock increases during a period spanning T units of time is*

$$\Pr\left[S_{t+T} > S_t\right] = \int_0^1 f(\theta)\left(N\left(\frac{\log\theta + (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) + \theta^{1-2\mu/\sigma^2} N\left(\frac{\log\theta - (\mu - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right) d\theta,$$

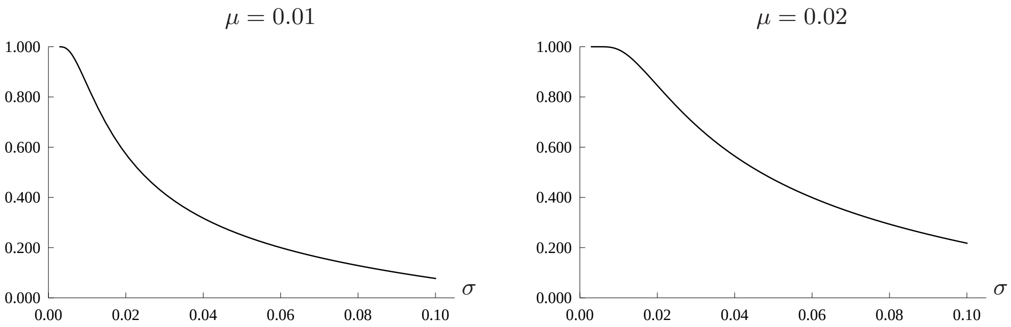
where $N(\cdot)$ is the cumulative distribution function for the standard normal distribution.

The two graphs in Figure 3 plot the unconditional probability that the annual change in the housing stock is positive, as a function of demand volatility, for the indicated levels of μ. Consistent with the fact that the probability in Proposition 10 does not depend on ε or η, the graphs show that the probability is the same for all three values of ε. In contrast to the behavior in Figure 2, here demand volatility has a very strong impact on development behavior, with greater volatility significantly reducing the probability that development occurs in any given year. Comparison of the two graphs also shows that development occurs more frequently when the average growth in demand is relatively high.

Finally, I consider the long-run average rate of development conditional on development actually occurring. Since the housing stock cannot fall, the unconditional average rate of development is

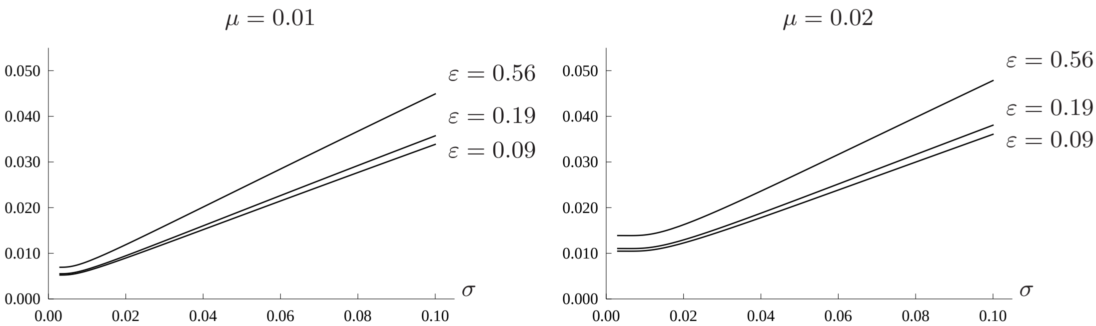
$$E[\log S_{t+T} - \log S_t] = E[\log S_{t+T} - \log S_t | S_{t+T} > S_t]\Pr\left[S_{t+T} > S_t\right],$$

Figure 3: Long-run probability of development



Each graph plots the unconditional probability that the annual change in the housing stock is positive for the indicated parameters. The probability does not depend on ε. The values of parameters not shown on the graphs are $\eta = 0.5$, $r = 0.03$, and $T = 1$.

Figure 4: Conditional average annual change in the log housing stock



The three curves in each graph plot the average annual change in the log housing stock, conditional on the change being positive, for the indicated parameters. The values of parameters not shown on the graphs are $\eta = 0.5$, $r = 0.03$, and $T = 1$.

which implies that

$$E[\log S_{t+T} - \log S_t | S_{t+T} > S_t] = \frac{E[\log S_{t+T} - \log S_t]}{\Pr\left[S_{t+T} > S_t\right]}.$$

This is easily calculated using the results of Propositions 9 and 10. The two graphs in Figure 4 show the conditional average rate of development, using the same format as Figure 2. When development occurs, it is more rapid when demand volatility is high. However, the average rate of growth in demand has only a minor effect on the rate of development when it occurs.

Combining the results of Figures 2–4 reveals an interesting picture of the determinants of growth in the housing stock.

- Higher long-run growth in demand makes development occur more frequently, but has little impact on the rate of development when it occurs. Overall, more rapid demand growth leads to higher average growth in the housing stock.

Table 3: The determinants of the long-run average rate of land development

	Probability of development	Rate of development when it occurs	Long-run average rate of development
Increased demand growth (μ)	Increased	Insignificant	Increased
Increased demand volatility (σ)	Decreased	Increased	Slightly decreased
Demand more elastic (η)	None	Increased	Increased
Land more heterogeneous (ε): Inelastic demand	None	Increased	Increased
Land more heterogeneous (ε): Elastic demand	None	Decreased	Decreased

- More volatile demand means that development will occur less frequently, but when it does occur it will be more rapid. Overall, the first effect dominates, but only slightly, so that the long-run rate of growth in the housing stock is relatively insensitive to the level of demand volatility.
- The price elasticity of housing demand has no effect on the frequency with which development occurs, but when development does occur it will be faster if demand is more elastic. Overall, the housing stock grows more rapidly if the demand for housing is more elastic.
- The degree of heterogeneity of undeveloped land has no effect on the frequency with which development occurs. When development does occur, greater land heterogeneity leads to faster development if demand is inelastic and slower demand if it is elastic. Overall, greater land heterogeneity leads to faster development with inelastic demand and slower demand with elastic demand.

These results are summarized in Table 3.

4 Conclusion

The model in this paper demonstrates that the option value of undeveloped land depends on the price elasticity of housing demand and the degree of cross-sectional variation in the quality of land. Relative to the standard open-city framework of Capozza and Helsley (1990) and other authors, competition amongst landowners reduces the option value. However, as long as undeveloped land is heterogeneous, it does not fall to zero. A special case of the model, calibrated to U.S. data, indicates that the value of waiting can drive an economically-significant wedge between equilibrium new house prices and the cost of developing additional land. This premium is greatest when housing demand is reasonably elastic, the interest rate is low, expected growth in demand is high, and land is highly heterogeneous. The last three requirements (and arguably the first one) describe the situation characterizing many of the cities where price–cost ratios have taken the highest values in recent years.[28]

[28]This is not meant to suggest that the high prices observed in some cities are consistent with fundamentals. However, as Shiller (2005) argues, speculative bubbles are typically precipitated by events that cause a rational

The qualitative results derived for the general model in Section 2 and the quantitative ones derived for the special case in Section 3 suggest various ways in which we might reexamine the empirical literature on housing supply. Much of the focus has been on understanding the variation in supply responses to high prices across the US. For example, Glaeser and Gyourko (2003) report that in 1999 the proportion of houses with prices exceeding construction costs by at least 40% was 96% in San Francisco (a growing coastal city), 65% in Phoenix (a growing inland one), and 20% in Detroit (a declining inland city).[29] Supply elasticities also show large variation, with Green et al. (2005) estimating figures of 0.14, 21.7, and 4.74 for San Francisco, Phoenix, and Detroit respectively.[30]

One explanation for price–cost ratios exceeding one and low supply elasticities is land-use regulation. This can prevent developers from constructing buildings of a sufficient density to equate marginal construction cost and price. Glaeser et al. (2005) present empirical evidence supporting the view that such regulation helps explain the sluggish supply response to high house prices. Green et al. (2005) and Saiz (2010) investigate the relationship between regulatory stringency and housing supply elasticity. They both report substantial variation in estimated elasticities across the US and find a strong negative relationship between supply elasticity and their measures of regulatory stringency, suggesting that regulation is an issue that cannot be dismissed.

The analysis in this paper suggests that the option value of land can be economically significant, so that high price–cost ratios can arise even without restrictive land-use regulation. It is therefore wrong to conclude that, because "[h]ome building is an enormously competitive industry with virtually no natural barriers to entry…, price markups over construction costs are a strong indication of artificial barriers to new construction" (Glaeser et al., 2005, p. 366). Rather, it is likely that high price–cost ratios can be explained by some combination of regulation and development timing options. If this is the case then only a part of the observed premium of house prices over construction costs can be attributed to land-use restrictions. Therefore, when Glaeser et al. (2005) claim that building restrictions mean that the "cost of housing is at least 50% more than it would be under a free-development policy" (p. 351), they may overstate the costs. Removing building restrictions would not remove the delay options embedded in undeveloped land, so house prices would not fall all the way to construction cost.

Both Green et al. (2005) and Saiz (2010) find factors additional to regulatory stringency that help predict supply elasticities: Green et al. show that supply elasticity and population density are negatively related, whereas Saiz shows that supply elasticity and the proportion of a city's area that is undevelopable (due to the presence of lakes, oceans, steeply-sloped land,

increase in the price of an asset, before irrational factors take over. The model in this paper suggests that fully-rational waiting by the owners of undeveloped land might be one of the precipitating factors that makes some housing markets especially prone to bubbles.

[29] In the same year, 54% of houses in Detroit had prices less than 90% of construction cost. The corresponding figures for San Francisco and Phoenix were negligible.

[30] Green et al. note that their supply elasticities for declining cities will likely be underestimates due to the fact that the durability of housing means that the housing stock will be reasonably unresponsive to declining demand.

and so on) are negatively related. The option value of land can help explain those parts of the price–cost ratio and supply elasticity not already explained by cross-sectional variations in regulatory stringency provided that the determinants of option value vary appropriately across the country. Green et al. and Saiz do not include proxies for demand growth or volatility when estimating the relationship between supply elasticity and regulatory stringency. It would be interesting to see if regulation still helps predict supply elasticity when these omitted variables are included.[31]

The results found by Saiz (2010) suggest that a city's physical geography can affect housing supply. The model presented in this paper offers one route towards examining this relationship. The special case in Section 3 assumes a tractable form of the cross-sectional variation in land quality, but there is no reason why a more realistic form could not be used, perhaps one based on city topography. For example, if the amount of housing services generated by a piece of land is related to commuting distance, then geographic data (similar to that used by Saiz) could be used to estimate the distribution of commuting distances for individual metropolitan areas and the results used to estimate the option value and supply elasticity implied by the model. One possible explanation for the relationship identified by Saiz is that as cities with greater topographical constraints grow, and the location of the city boundary moves, the desirability of land on the boundary falls more rapidly than in less constrained cities with the same growth rate, making the development-timing options embedded in undeveloped land more valuable in more constrained cities.

Finally, the results in this paper have implications for the empirical literature testing the predictions of real options models using data from real estate markets. In particular, Table 3 shows that the model specification should be sufficiently flexible that changes in explanatory variables can move the unconditional and conditional average rates of development in opposite directions. This requirement eliminates some popular model specifications. Consider, for example, Schwartz and Torous (2007), who analyze quarterly data on the number of office buildings started in various U.S. metropolitan areas. They adopt a Poisson regression model, which has the density function

$$f(n|\mathbf{x}) = \frac{e^{-\nu(\mathbf{x})}(\nu(\mathbf{x}))^n}{n!},$$

where $\nu(\mathbf{x}) = \exp(\mathbf{a}'\mathbf{x})$ and $\mathbf{x}$ is a vector of explanatory variables. With this specification, the probability of zero starts is a decreasing function of ν, the unconditional expected number of starts is increasing in ν, and the expected number of starts conditional on the number being positive is also increasing in ν. This specification is therefore too restrictive to test the predictions developed in this paper, because a change in the level of any explanatory variable will always move the conditional and unconditional expected values in the same direction. For example, if the coefficient on volatility is negative (which Schwartz and Torous find it to be) then greater volatility makes it more likely that zero starts are observed and reduces the unconditional

[31] Note that equation (13), which gives the static supply function for the special case of the model in Section 3, shows that supply is an increasing function of β and therefore a decreasing function of μ and σ.

expected number of starts, but also reduces the expected number of starts conditional on the number being positive. The first two results are consistent with the predictions here, but the third is not.[32]

The Poisson hurdle model introduced by Mullahy (1986) addresses some of these concerns. This model separates the probability of investment occurring from the amount of investment that occurs. The density function is

$$f(n|\mathbf{x}) = \begin{cases} e^{-\lambda(\mathbf{x})} & \text{if } n = 0, \\ \frac{e^{-\nu(\mathbf{x})}(\nu(\mathbf{x}))^n}{n!} \cdot \frac{1-e^{-\lambda(\mathbf{x})}}{1-e^{-\nu(\mathbf{x})}}, & \text{if } n > 0, \end{cases}$$

where $\nu(\mathbf{x}) = \exp(\mathbf{a}'\mathbf{x})$ and $\lambda(\mathbf{x}) = \exp(\mathbf{b}'\mathbf{x})$. It reduces to the standard Poisson model when $\mathbf{a} = \mathbf{b}$. However, in the general case the probability of non-zero investment equals $1 - e^{-\lambda(\mathbf{x})}$ and the conditional expected number of starts equals

$$E[n|n > 0, \mathbf{x}] = \frac{\nu(\mathbf{x})}{1 - e^{-\nu(\mathbf{x})}}.$$

Thus the parameter vector **b** determines the probability that there are any new starts, whereas the parameter vector **a** determines the average number of starts, conditional on that number being positive. The results in Table 3 suggest that the coefficient on volatility in **a** will be positive, whereas the one in **b** will be negative.

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[32] Schwartz and Torous (2007) report numerous quarters when there were no new starts of office buildings in their sample, so the issues discussed here are relevant. Over 50% of their sample involved no more than two starts of investment-grade properties per quarter, while the average number of starts was 4.9 and the maximum was 37.

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Appendix: Proofs

Proof of Proposition 1

The social planner's problem is mathematically equivalent to the capacity-choice problem of a monopolist with a production function exhibiting diminishing returns to scale analyzed on pp. 360–367 of Dixit and Pindyck (1994). All that changes is that the monopolist's profit flow $X_t\Psi(H(S_t))H(S_t)$ is replaced by the planner's total surplus flow

$$X_t \int_0^{S_t} \Psi(H(S))H'(S)dS.$$

The key is that both flow functions are concave in S_t. Rather than repeat their proof here, I simply refer the reader to Dixit and Pindyck (1994).

Proof of Proposition 2

I begin by solving for $V(S, X; \bar{S})$ over the region where $X < \hat{X}(S)$. In this region the housing stock will be constant over the next dt units of time, so that

$$V(S, X; \bar{S}) = X\Phi(S)h(\bar{S})dt + e^{-r\,dt}E[V(S, X + dX; \bar{S})].$$

This implies that V satisfies

$$0 = \frac{1}{2}\sigma^2 X^2 V_{XX} + \mu X V_X - rV + X\Phi(S)h(\bar{S}).$$

If $X = 0$ then all future rent equals zero, which implies that $V(S, 0; \bar{S}) = 0$ for all S. When combined with the differential equation, this implies that

$$V(S, X; \bar{S}) = \frac{X\Phi(S)h(\bar{S})}{r - \mu} + B(S)X^\beta, \quad X < \hat{X}(S),$$

for some function $B(S)$ that is still to be determined.

Now I solve for $V(S, X; \bar{S})$ over the region where $X > \hat{X}(S)$. In this region, the stock of developed land is immediately increased to the level $\hat{S}(X) \equiv \hat{X}^{-1}(X)$. Since there is no immediate cash flow for the owner of already-developed land (although future rents will be affected), the market value of a unit of developed land equals

$$V(S, X; \bar{S}) = V(\hat{S}(X), X; \bar{S}), \quad X > \hat{X}(S).$$

Continuity of V along the boundary then implies that

$$V(S, X; \bar{S}) = \frac{X\Phi(\hat{S}(X))h(\bar{S})}{r-\mu} + B(\hat{S}(X))X^{\beta}, \quad X \geq \hat{X}(S).$$

Because the planner is using a policy of barrier control, $\frac{\partial V}{\partial X}$ will be continuous along the boundary between these two regions. Differentiating the two expressions for $V(S, X; \bar{S})$ with respect to X shows that

$$V_X(S, X; \bar{S}) = \frac{\Phi(S)h(\bar{S})}{r-\mu} + \beta B(S)X^{\beta-1}, \quad X < \hat{X}(S),$$

and

$$\begin{aligned} V_X(S, X; \bar{S}) \quad &= \quad \frac{\Phi(\hat{S}(X))h(\bar{S})}{r-\mu} + \frac{X\Phi'(\hat{S}(X))h(\bar{S})\hat{S}'(X)}{r-\mu} \\ &\quad + \beta B(\hat{S}(X))X^{\beta-1} + B'(\hat{S}(X))\hat{S}'(X)X^{\beta}, \quad X \geq \hat{X}(S). \end{aligned}$$

Continuity of $V_X(S, X; \bar{S})$ along the curve where $X = \hat{X}(S)$ implies that

$$B'(S) = \frac{-\Phi'(S)h(\bar{S})}{(r-\mu)(\hat{X}(S))^{\beta-1}}.$$

Solving this differential equation, together with the boundary condition $B(S) \to 0$ as $S \to \infty$, completes the calculation of the price of developed land.[33]

Proof of Proposition 3

The price of the marginal piece of developed land equals

$$V(S_t, X_t; S_t) = \frac{X_t h(S_t)}{r-\mu}\left(\Phi(S_t) + \int_{S_t}^{\infty} \Phi'(S)\left(\frac{X_t}{\hat{X}(S)}\right)^{\beta-1} dS\right),$$

which reduces to

$$V(S_t, X_t; S_t) = \frac{X_t h(S_t)}{r-\mu}\left(\Phi(S_t) + \left(\frac{X_t}{\hat{X}(S_t)}\right)^{\beta-1} \int_{S_t}^{\infty} \Phi'(S)\left(\frac{\Phi(S)h(S)}{\Phi(S_t)h(S_t)}\right)^{\beta-1} dS\right).$$

Applying integration by parts shows that

$$\begin{aligned} V(S_t, X_t; S_t) \quad &= \quad \frac{X_t\Phi(S_t)h(S_t)}{r-\mu}\left(1 - \frac{1}{\beta}\left(\frac{X_t}{\hat{X}(S_t)}\right)^{\beta-1}\right. \\ &\quad \left. + \frac{\beta-1}{\beta}\left(\frac{X_t}{\hat{X}(S_t)}\right)^{\beta-1} \int_{S_t}^{\infty} \left(\frac{\Phi(S)}{\Phi(S_t)}\right)^{\beta}\left(\frac{h(S)}{h(S_t)}\right)^{\beta-1}\left(\frac{-h'(S)}{h(S)}\right) dS\right). \end{aligned}$$

[33]If the housing stock is very large then there will be no additional development for a considerable period, so that S can be regarded as constant and the house price equal to $X\Phi(S)h(\bar{S})/(r-\mu)$. This holds only if $B(S) \to 0$ as $S \to \infty$.

It follows that

$$\begin{aligned}\frac{\partial V(S_t,X_t;S_t)}{\partial X_t} &= \frac{\Phi(S_t)h(S_t)}{r-\mu}\left(1-\left(\frac{X_t}{\hat{X}(S_t)}\right)^{\beta-1}\right. \\ &\quad \left. +(\beta-1)\left(\frac{X_t}{\hat{X}(S_t)}\right)^{\beta-1}\int_{S_t}^{\infty}\left(\frac{\Phi(S)}{\Phi(S_t)}\right)^{\beta}\left(\frac{h(S)}{h(S_t)}\right)^{\beta-1}\left(\frac{-h'(S)}{h(S)}\right)dS\right),\end{aligned}$$

which is clearly positive whenever $X_t < \hat{X}(S_t)$. It follows that $V(S_t,X_t;S_t) < V(S_t,\hat{X}(S_t);S_t)$ whenever $X_t < \hat{X}(S_t)$. The proof follows from noting that

$$V(S_t,\hat{X}(S_t);S_t) = C\left(1+\int_{S_t}^{\infty}\left(\frac{\Phi(S)}{\Phi(S_t)}\right)^{\beta}\left(\frac{h(S)}{h(S_t)}\right)^{\beta-1}\left(\frac{-h'(S)}{h(S)}\right)dS\right).$$

Proof of Proposition 4

The piece of land of type $\bar{S}$ will be developed as soon as the state variable equals $\hat{X}(\bar{S})$, implying a payoff of

$$V(\bar{S},\hat{X}(\bar{S});\bar{S}) - C = C\left(\frac{V(\bar{S},\hat{X}(\bar{S});\bar{S})}{C}-1\right) = C\int_{\bar{S}}^{\infty}\left(\frac{\Phi(S)}{\Phi(\bar{S})}\right)^{\beta}\left(\frac{h(S)}{h(\bar{S})}\right)^{\beta-1}\left(\frac{-h'(S)}{h(S)}\right)dS,$$

where I have used Proposition 3. Since this payoff will be received as soon as X_t reaches $\hat{X}(\bar{S})$ from below, it follows that the present value of the payoff is the product of the discount factor $(X_t/\hat{X}(\bar{S}))^{\beta}$ and the certain payoff, which gives the expression in equation (8).

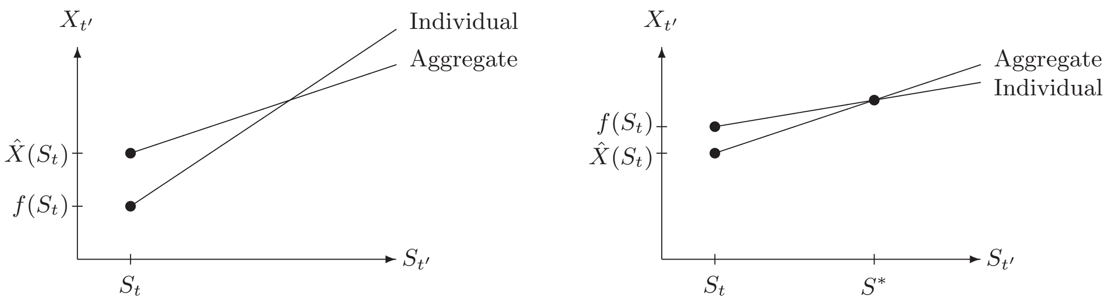
Proof of Proposition 5

I show that an individual landowner cannot do better than follow the development policy in the proposition, given that all other landowners are following the policy described there. The proof begins by describing the evolution of the housing stock, and then uses this to determine the house-price process, which the individual landowner treats as exogenous. I then derive an optimal development policy for the individual landowner, which coincides with the one in the proposition.

First, suppose that all landowners follow the policy described in the proposition and that the housing stock is initially zero: $S_0 = 0$. Then an arbitrary piece of undeveloped land of type $\bar{S}$ will be developed at date 0 if and only if $X_0 \geq \hat{X}(\bar{S})$. Thus, the housing stock will be immediately raised to the level $\hat{S}(X_0) = \hat{X}^{-1}(X_0)$, with all pieces of land of type $\bar{S} \leq \hat{S}(X_0)$ being developed at date 0.

In order to completely describe the evolution of the housing stock, I need to consider what happens at an arbitrary future date t. Suppose that at date t, all pieces of land of type $\bar{S} \leq S_t$ have been developed, and all other pieces of land are undeveloped. There are two possibilities to consider. In the first case, $X_t \leq \hat{X}(S_t)$. An arbitrary piece of undeveloped land has type $\bar{S}$ for some $\bar{S} > S_t$, so that $X_t \leq \hat{X}(S_t) < \hat{X}(\bar{S})$. The owner of this piece of land therefore waits, implying that no development occurs immediately if $X_t \leq \hat{X}(S_t)$. In the second case,

Figure A.1: Possible development policies for an individual landowner



$X_t > \hat{X}(S_t)$. All pieces of undeveloped land for which $\hat{X}(\bar{S}) \leq X_t$ will be developed immediately. That is, all pieces of land of types up to and including $\hat{S}(X_t)$ will be developed immediately. It follows that if all landowners adopt the prescribed development policy then the housing stock will evolve according to the development policy described in Proposition 1, in which case the house price is given in Proposition 2.

Now consider the problem facing the owner of a piece of land of type $\bar{S}$ at date t. Suppose he follows a policy of developing his piece of land as soon as $X_{t'} \geq f(S_{t'})$ for some continuous function f. If $f(S_t) \leq \hat{X}(S_t)$ then—because his individual development threshold is currently below the aggregate one—he will develop his land before any other land is developed. Consequently, he actually develops his land as soon as $X_{t'} \geq f(S_t)$, so that the development threshold can be treated as a constant (from the perspective of date t). This situation is shown in the left-hand graph in Figure A.1. If $f(S_t) > \hat{X}(S_t)$ then—because his individual development threshold is currently above the aggregate one—he actually develops his land as soon as $X_{t'} \geq f(S^*)$ where S^* is the smallest number greater than S_t such that $f(S^*) = \hat{X}(S^*)$. Once again, the development threshold can be treated as a constant. This situation is shown in the right-hand graph in Figure A.1. I now find a value-maximizing level of this constant.

If the landowner chooses a development threshold $\bar{X} \leq \hat{X}(S_t)$, then the housing stock will still equal S_t when his land is eventually developed, so that his development payoff will be $V(S_t, \bar{X}; \bar{S}) - C$. The market value of his land at date t therefore equals

$$F(S_t, X_t; \bar{X}) = \big(V(S_t, \bar{X}; \bar{S}) - C\big) \left(\frac{X_t}{\bar{X}}\right)^{\beta},$$

so that

$$\begin{aligned} \frac{\partial F(S_t, X_t; \bar{X})}{\partial \bar{X}} &= \left(V_X(S_t, \bar{X}; \bar{S}) - \frac{\beta}{\bar{X}}\big(V(S_t, \bar{X}; \bar{S}) - C\big)\right)\left(\frac{X_t}{\bar{X}}\right)^{\beta} \\ &= \left(\frac{\beta C}{\bar{X}} - \frac{(\beta - 1)h(\bar{S})\Phi(S_t)}{r - \mu}\right)\left(\frac{X_t}{\bar{X}}\right)^{\beta} \\ \frac{\partial F(S_t, X_t; \bar{X})}{\partial \bar{X}} &= \beta C \left(\frac{1}{\bar{X}} - \frac{1}{\hat{X}(S_t)} \cdot \frac{h(\bar{S})}{h(S_t)}\right)\left(\frac{X_t}{\bar{X}}\right)^{\beta}. \end{aligned} \tag{A.1}$$

However, if the landowner chooses $\bar{X} \geq \hat{X}(S_t)$, then the housing stock will have increased to $\hat{S}(\bar{X})$ by the time the land is developed. It follows that the landowner's payoff will be $V(\hat{S}(\bar{X}), \bar{X}; \bar{S}) - C$ and that the market value of the land at date t equals

$$F(S_t, X_t; \bar{X}) = \Big(V(\hat{S}(\bar{X}), \bar{X}; \bar{S}) - C\Big) \left(\frac{X_t}{\bar{X}}\right)^{\beta}.$$

In this case

$$\frac{\partial F(S_t, X_t; \bar{X})}{\partial \bar{X}} = \left(V_S(\hat{S}(\bar{X}), \bar{X}; \bar{S})\hat{S}'(\bar{X}) + V_X(\hat{S}(\bar{X}), \bar{X}; \bar{S}) - \frac{\beta}{\bar{X}}\Big(V(\hat{S}(\bar{X}), \bar{X}; \bar{S}) - C\Big)\right) \left(\frac{X_t}{\bar{X}}\right)^{\beta}.$$

Notice that

$$V_S(S, \bar{X}; \bar{S}) = \frac{\bar{X}\Phi'(S)h(\bar{S})}{r-\mu} \left(1 - \left(\frac{\bar{X}}{\hat{X}(S)}\right)^{\beta-1}\right)$$

whenever $\bar{X} \leq \hat{X}(S)$, and that it equals zero when $S = \hat{S}(\bar{X})$. Therefore

$$\begin{aligned} \frac{\partial F(S_t, X_t; \bar{X})}{\partial \bar{X}} &= \left(V_X(\hat{S}(\bar{X}), \bar{X}; \bar{S}) - \frac{\beta}{\bar{X}}\Big(V(\hat{S}(\bar{X}), \bar{X}; \bar{S}) - C\Big)\right) \left(\frac{X_t}{\bar{X}}\right)^{\beta} \\ &= \left(\frac{\beta C}{\bar{X}} - (\beta-1)\frac{h(\bar{S})\Phi(\hat{S}(\bar{X}))}{r-\mu}\right) \left(\frac{X_t}{\bar{X}}\right)^{\beta} \\ \frac{\partial F(S_t, X_t; \bar{X})}{\partial \bar{X}} &= \frac{(\beta-1)\Phi(\hat{S}(\bar{X}))}{r-\mu}\Big(h(\hat{S}(\bar{X})) - h(\bar{S})\Big) \left(\frac{X_t}{\bar{X}}\right)^{\beta}. \end{aligned} \tag{A.2}$$

Since the land is currently undeveloped, it cannot have the potential to generate more housing services than the marginal piece of developed land (that is $h(\bar{S}) \leq h(S_t)$). Equation (A.1) therefore shows that F is increasing in $\bar{X}$ for all $\bar{X} < \hat{X}(S_t)$, while equation (A.2) shows that the landowner should choose $\bar{X} = \hat{S}^{-1}(\bar{S}) = \hat{X}(\bar{S})$ in this situation, since this ensures that $h(\hat{S}(\bar{X})) = h(\bar{S})$ and $\partial F / \partial \bar{X} = 0$.[34] Therefore, given that all other landowners are following the development policy described in the proposition, an arbitrary individual landowner cannot do better than follow that policy as well.

Proof of Proposition 6

It is easily shown that

$$\frac{V(S, \hat{X}(S); S)}{C} = \frac{h(S)}{C} \cdot \frac{\hat{X}(S)S^{-(1-\varepsilon)/\eta}}{r-\mu} \cdot \frac{(\beta-1)(1-\varepsilon+\varepsilon\eta)}{\beta(1-\varepsilon)+(\beta-1)\varepsilon\eta} = \frac{1}{1 - \frac{\varepsilon\eta}{\beta(1-\varepsilon+\varepsilon\eta)}}.$$

Proof of Proposition 7

The proof of Proposition 2 applies for *any* policy of barrier control. If, for example, the minimum amount of land development occurs to prevent X_t from climbing above $\hat{Y}(S_t)$, for some function

[34] If the landowner were ever to find himself in a situation where his land had the potential to generate more housing services than the marginal piece of developed land (that is, $h(\bar{S}) > h(S_t)$) then the situation is slightly different: equation (A.2) shows that F is decreasing in $\bar{X}$ for all $\bar{X} > \hat{X}(S_t)$, since $h(\hat{S}(\bar{X})) \leq h(S_t) < h(\bar{S})$, while equation (A.1) shows that the landowner should choose $\bar{X} = \hat{X}(S_t)h(S_t)/h(\bar{S}) < \hat{X}(S_t)$.

$\hat{Y}$, then developed land of type $\bar{S}$ will be worth

$$V(S_t, X_t; \bar{S}) = \frac{X_t \Phi(S_t) h(\bar{S})}{r-\mu} \left(1 + \int_{S_t}^{\infty} \frac{\Phi'(S)}{\Phi(S_t)} \left(\frac{X_t}{\hat{Y}(S)} \right)^{\beta-1} dS \right) \tag{A.3}$$

for all $X_t \leq \hat{Y}(S_t)$. Immediately after this piece of land is developed, it is worth

$$V(\bar{S}, \hat{Y}(\bar{S}); \bar{S}) = \frac{\hat{Y}(\bar{S}) \Phi(\bar{S}) h(\bar{S})}{r-\mu} \left(1 + \int_{\bar{S}}^{\infty} \frac{\Phi'(S)}{\Phi(\bar{S})} \left(\frac{\hat{Y}(\bar{S})}{\hat{Y}(S)} \right)^{\beta-1} dS \right).$$

In the special case where $\Phi(S) = S^{-(1-\varepsilon)/\eta}$, $h(S) = S^{-\varepsilon}$, and $\hat{Y}(S) = AS^{\varepsilon+(1-\varepsilon)/\eta}$, for some constant A, this land value simplifies to

$$\begin{aligned} V(\bar{S}, \hat{Y}(\bar{S}); \bar{S}) &= \frac{A}{r-\mu} \left(1 - \frac{1-\varepsilon}{\eta} \int_{\bar{S}}^{\infty} \left(\frac{S}{\bar{S}} \right)^{-\beta(1-\varepsilon)/\eta - (\beta-1)\varepsilon - 1} \frac{dS}{\bar{S}} \right) \\ &= \frac{A}{r-\mu} \left(1 - \frac{1-\varepsilon}{\beta(1-\varepsilon) + (\beta-1)\varepsilon\eta} \right). \end{aligned}$$

This equals the development cost provided that

$$A = \left(\frac{1}{1 + \frac{\varepsilon\eta}{\beta(1-\varepsilon)+(\beta-1)\varepsilon\eta}} \right) (r-\mu) \left(\frac{\beta}{\beta-1} \right) C.$$

That is, if landowners develop their land whenever the house price is greater than or equal to the development cost then the housing stock is determined by the development threshold

$$\hat{X}^*(S) = \left(\frac{1}{1 + \frac{\varepsilon\eta}{\beta(1-\varepsilon)+(\beta-1)\varepsilon\eta}} \right) (r-\mu) \left(\frac{\beta}{\beta-1} \right) C S^{\varepsilon+(1-\varepsilon)/\eta}.$$

The house price is given by (A.3) with $\hat{Y}(S)$ replaced by $\hat{X}^*(S)$.

Proof of Proposition 8

It is easily shown that

$$\frac{S_t^*}{S_t} = \left(1 + \frac{\varepsilon\eta}{\beta(1-\varepsilon) + (\beta-1)\varepsilon\eta} \right)^{1/(\varepsilon+(1-\varepsilon)/\eta)}.$$

When delay options are exercised optimally, the house price for this level of demand equals

$$V(S_t, \hat{X}(S_t); \bar{S}) = h(\bar{S}) \frac{\hat{X}(S_t) S_t^{-(1-\varepsilon)/\eta}}{r-\mu} \left(1 - \frac{1-\varepsilon}{\beta(1-\varepsilon) + (\beta-1)\varepsilon\eta} \right),$$

whereas it equals

$$V^*(S_t^*, \hat{X}(S_t); \bar{S}) = h(\bar{S}) \frac{\hat{X}(S_t) (S_t^*)^{-(1-\varepsilon)/\eta}}{r-\mu} \left(1 - \frac{1-\varepsilon}{\beta(1-\varepsilon) + (\beta-1)\varepsilon\eta} \right)$$

when delay options are ignored. It follows that the proportion of the equilibrium house price attributable to development timing options is

$$\frac{V - V^*}{V} = 1 - \left(\frac{S_t^*}{S_t} \right)^{-(1-\varepsilon)/\eta} = 1 - \left(1 + \frac{\varepsilon\eta}{\beta(1-\varepsilon) + (\beta-1)\varepsilon\eta} \right)^{-(1-\varepsilon)/(1-\varepsilon+\varepsilon\eta)}.$$

Proof of Lemma 1

Suppose that θ follows a geometric Brownian motion with an upper reflecting barrier at B, drift of μ, and volatility of σ. Then $z = \log\theta$ follows an arithmetic Brownian motion with an upper reflecting barrier at $\log B$, drift of $\nu = \mu - \frac{1}{2}\sigma^2$, and volatility of $\phi = \sigma$. From p. 61 of Dixit (1993), provided $\nu > 0$, the long-run distribution of z has density function[35]

$$g(z) = \frac{2\nu}{\phi^2} \exp\left(\frac{2\nu(z - \log B)}{\phi^2}\right).$$

It follows that θ has density function

$$f(\theta) = \frac{g(\log\theta)}{\theta} = \frac{2\nu}{\phi^2 B}\left(\frac{\theta}{B}\right)^{2\nu/\phi^2 - 1} = \frac{2\mu - \sigma^2}{\sigma^2 B}\left(\frac{\theta}{B}\right)^{2\mu/\sigma^2 - 2}.$$

Proof of Proposition 9

The proof follows the lines of a similar result on p. 373 of Dixit and Pindyck (1994). Let $\theta_t = X_t/\hat{X}(S_t)$. The process for $\log\theta_t$ is approximated by a discrete random walk in which time is represented by periods each lasting dt units of time and $\log\theta_t$ increases by $\sigma\sqrt{dt}$ with probability

$$p = \frac{1}{2} + \frac{(\mu - \frac{1}{2}\sigma^2)\sqrt{dt}}{2\sigma}$$

and otherwise falls by the same amount. If $\log\theta_t < 1 - \sigma\sqrt{dt}$ then even an up move will keep the state variable below the development threshold. Therefore, the only situation in which development will occur within the next dt units of time is if θ_t is within one step (of length $\sigma\sqrt{dt}$) of 1 and the next move is up. The long-run probability that θ_t is within one step of the upper barrier is approximately equal to $f(1)\sigma\sqrt{dt}$, where $f(1) = 2\mu/\sigma^2 - 1$. Since the next move is up with probability p, the long-run probability that the housing stock increases over any short interval lasting dt units of time is

$$pf(1)\sigma\sqrt{dt} = \left(\frac{1}{2} + \frac{(\mu - \frac{1}{2}\sigma^2)\sqrt{dt}}{2\sigma}\right)\left(\frac{2\mu}{\sigma^2} - 1\right)\sigma\sqrt{dt}.$$

To complete the proof, note that if $X_t/\hat{X}(S_t) = 1$ and $\log X_t$ increases by $\sigma\sqrt{dt}$ then the log housing stock must increase by $\eta\sigma\sqrt{dt}/(1-\varepsilon+\varepsilon\eta)$. The long-run average change in the housing stock over any interval lasting dt units of time is therefore equal to

$$\left(\frac{1}{2} + \frac{(\mu - \frac{1}{2}\sigma^2)\sqrt{dt}}{2\sigma}\right)\left(\frac{2\mu}{\sigma^2} - 1\right)\sigma\sqrt{dt} \cdot \frac{\eta\sigma\sqrt{dt}}{1-\varepsilon+\varepsilon\eta} \approx \frac{\left(\mu - \frac{\sigma^2}{2}\right)\eta}{1-\varepsilon+\varepsilon\eta}dt.$$

Thus, the long-run average growth rate in the housing stock is $(\mu - \frac{\sigma^2}{2})\eta/(1-\varepsilon+\varepsilon\eta)$, as required.

[35] If $\nu < 0$ then the distribution is degenerate.

Proof of Proposition 10

Investment occurs between dates t and $t+T$ if and only if $X_{t'} > \hat{X}(S_t)$ for some t' in this interval. When viewed from the perspective of date t, the probability of investment occurring is therefore

$$\begin{aligned}
\Pr\left[\max_{t'\in[t,t+T]} X_{t'} > \hat{X}(S_t)\right] &= 1-\Pr\left[\max_{t'\in[t,t+T]} X_{t'} \leq \hat{X}(S_t)\right] \\
&= 1-\Pr\left[\log\left(\max_{t'\in[t,t+T]} X_{t'}/X_t\right) \leq \log\left(\hat{X}(S_t)/X_t\right)\right] \\
&= 1-N\left(\frac{\log\left(\hat{X}(S_t)/X_t\right)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&\quad +\left(\frac{\hat{X}(S_t)}{X_t}\right)^{2\mu/\sigma^2-1} N\left(\frac{-\log\left(\hat{X}(S_t)/X_t\right)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&= N\left(\frac{\log\left(X_t/\hat{X}(S_t)\right)+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right) \\
&\quad +\left(\frac{X_t}{\hat{X}(S_t)}\right)^{1-2\mu/\sigma^2} N\left(\frac{\log\left(X_t/\hat{X}(S_t)\right)-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right),
\end{aligned}$$

where I have used Lemma 3 of Conze and Viswanathan (1991). Averaging over all possible values of $X_t/\hat{X}(S_t)$ shows that the unconditional probability equals

$$\Pr[S_{t+T} > S_t] = \int_0^1 f(\theta)\left(N\left(\frac{\log\theta+(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)+\theta^{1-2\mu/\sigma^2}N\left(\frac{\log\theta-(\mu-\frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}\right)\right)d\theta.$$